Computing Optimized Nonlinear Sliding Surfaces

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***Abstract*— In this paper, we have concentrated on real systems consisting of structural uncertainties and affected by external disturbances. In this regard, Sliding Mode Control (S.M.C.) is utilized. To decrease energy consumption, arising from chattering phenomenon, a smooth switch has been used in design procedure. Consequently, sliding equation will play a dominant controlling role in its neighborhood. The converging property of sliding motion towards the origin is a challenging issue. In this article we present a new method to prove the stability of the sliding phase which means, state trajectories on the sliding surface move toward the origin. At the beginning, the equivalent control method is reestablished such that makes this purpose accessible. The modification bounds the sliding equation to a converging set. Then to improve main factors of closed loop system, such as, transient behavior, energy consumption and the domain of attraction, the optimal control theory is used to compute the optimized sliding surface in the stabilizing set. Generally, desired surface has nonlinear terms. Finally, we propose an elaborate algorithm for computing optimized nonlinear surfaces. The designed controller is applied to a flexible–link setup. Simulation results show the efficiency of the proposed approach.**

I. Introduction

The conventional S.M.C. has been developed for linear surfaces; yet there are also a number of design approaches with non linear surfaces. The equivalent method may be used to design sliding mode controllers, [1]. The computed signal forces state trajectory to reach sliding motion, but it does not support their motion towards the origin on the selected surface. In the sliding phase the surface equation has the most important role to result in the desired performance. In the conventional approach the sliding equation usually proposed as a linear combination of state trajectories, [2].

In this paper, we present a new algorithm to design an optimal stable non linear sliding surface for affine systems. The obtained surface also decreases energy consumption, improves the transient behavior and expands the region of attraction. These factors are affected by the sliding surface. Using of a smooth switch makes this process easier.

Using our method, the order of design steps in the conventional method is changed. In the current S.M.C., the designer assumes a sliding surface without concerning the stability of the sliding motion. Then, the equivalent control approach is used to compute the control signal. However, in this paper, we obtain a stabilizing control signal for nominal system. By the nominal system, we mean a mathematical model of the real system without any uncertainty or external disturbances. This control signal is used instead of the equivalent control signal. This kind of selection of the equivalent control signal changes only the value of the switch gain. Finally, optimal sliding equation is computed by applying nonlinear optimal control to the modified structure.The modification process is well directed and mathematically supported.

To further elaborate the proposed scheme, the paper has been organized as follows: Section II presents the basis of the conventional S.M.C. (Sliding Mode Control). In Section III, a new method is presented in order to compute the stabilizing equivalent signal. A control law which results in a converging sliding motion will be obtained in this section. The required elements of the theory of optimal control has been reviewed in Section IV. Also the taken strategy to coincide optimal control with the modified sliding mode is declared in this section. Section V presents the flexible–link robot example, and the simulations to verify the effectiveness of the proposed approach.

II. Sliding mode controller design

State space equations of the affine systems may be explained in the form:

$$\dot{x} = f(x) + g(x)u + \varpi \tag{1}$$
$$x, \varpi \in \mathbb{R}^n, f \in \mathbb{R}^n, g \in \mathbb{R}^{n\times m}, u \in \mathbb{R}^m$$

The nominal dynamics may be viewed as

$$\dot{x} = f(x) + g(x)u. \tag{2}$$

To design the control signal based on the conventional S.M.C. theory, first a sliding manifold is considered. To find the equivalent control law, time derivative of this surface should equal zero, [1], [4]. In this connection, equation (2) is used.

$$\begin{aligned} \dot{s}(x) &= \frac{\partial s}{\partial x}\frac{\partial x}{\partial t} \\ \dot{s}(x) &= U(x)f(x) + U(x)g(x)u \\ U_{ij} &\triangleq \frac{\partial s_i}{\partial x_j}, \quad i = 1\cdots\ m,\ j = 1\cdots\ n \end{aligned} \tag{3}$$

Assuming $\det\Big(U(x)g(x)\Big) \neq 0, \forall x \in \mathbb{R}^n$, then u_{eq} is obtained by setting (3) equal to zero.

$$u_{\mathrm{eq}} = -\Big(U(x)g(x)\Big)^{-1}U(x)f(x) \tag{4}$$

A sign part will be added to the control signal to show robustness against disturbance term (ϖ) in real dynamics.

$$\begin{aligned} u &= u_{\mathrm{eq}} - u_{\mathrm{s}} \\ u_{\mathrm{s}} &= k'\mathrm{sign}(s), \quad k' \triangleq \Big(U(x)g(x)\Big)^{-1}k \\ k &= \mathrm{Diag}\{k_1\ \cdots\ k_m\} \end{aligned} \tag{5}$$

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In order to prove the convergence of the phase trajectories to the sliding manifold, consider a Lyapunov function in the form

$$V = \frac{1}{2}s^{\mathrm{T}}(x)s(x). \tag{6}$$

It is realized that,

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial s}\frac{\partial s}{\partial t} \\ \dot{V} &= s^{\mathrm{T}}U\Big(f + gu + \varpi\Big) \end{aligned} \tag{7}$$

By substituting u from (5) in (7) and since $U(x)g(x)$ is a nonsingular matrix,

$$\begin{aligned} \dot{V} &= -s^{\mathrm{T}}k\mathrm{sign}(s) + s^{\mathrm{T}}Ud \\ &= -\sum_{i=1}^{m} k_i|s_i| + \sum_{i=1}^{m} q_i s_i < 0, \quad q_i \triangleq \frac{\partial s_i}{\partial x}\varpi \end{aligned} \tag{8}$$

Negative definiteness of $\dot{V}$ will be satisfied, if k_i fulfills the following condition.

$$k_i = |q_i| + \epsilon_i, \quad \epsilon_i > 0 \text{ for } i = 1 \cdots m \tag{9}$$

This shows the stability of reaching phase.

After state trajectories reach to $s = 0$, model order decreases by m. The stability of remaining dynamics whose order is $n - m$, is associated with the sliding manifold. Usually, in canonical companion systems, selecting linear surfaces and fulfillment of Hurwitz criterion will result in the sliding motion stability. If it is not possible to transform the model to companion form, the stability of the sliding mode for any case is supposed to be fulfilled. This is an important disadvantage in the conventional S.M.C. method, which in this paper we present a new approach to remove this drawback.

The other disadvantage is the chattering phenomenon. Using an approximation of the sign function in a thin boundary layer neighboring the switching surface may solve this problem. Consequently, sigmoid function instead of a sign function is selected, [8], [3]. The approximation reads as follows.

$$u_{\mathrm{s}} = k' \tanh(s) \tag{10}$$

III. Modification of S.M.C. to obtain a converging sliding motion

As it was noted, satisfying the convergence of the sliding motion towards the origin is a basic step to design the controller. There is not a general way to prove the convergence of this motion. This problem is more critical when a nonlinear sliding surface is selected. The equivalent control method is a convenient method to design the control signal. However, there has been no way to guarantee the appropriateness of the sliding surface, in terms of stability of the closed loop system. The design strategy is almost based on a try and error algorithm to find an appropriate surface. An inaccurate selection of this manifold with adding nonlinear terms may cause new equilibrium points added in the closed loop system. To describe this phenomenon, we present an example regarding the control of an inverted pendulum.

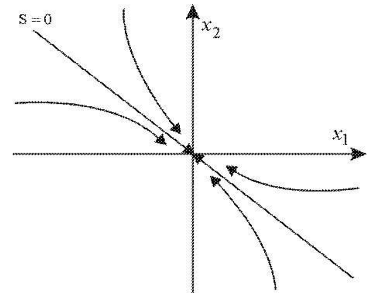

Fig. 1. Phase plane of inverted pendulum for surface s_1.

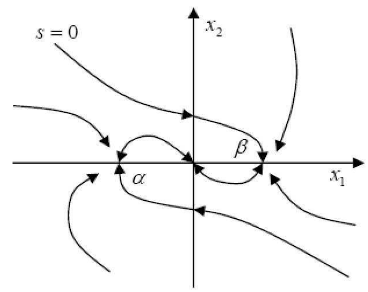

Fig. 2. Phase plane of inverted pendulum for surface s_2. Two new equilibrium points, α and β, are entered in the closed loop system.

Example 1: Dynamics of an inverted pendulum in the presence of viscous friction is given as:

$$\ddot{\theta} = \sin(\theta) - \dot{\theta} + \tau$$

State space realization of this system is obtained by defining $x_1 = \theta$, $x_2 = \dot{\theta}$ and $u = \tau$. Sliding mode control structure is deployed here.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) - x_2 + u \\ u &= u_{\mathrm{eq}} - u_{\mathrm{s}}, \qquad u_{\mathrm{eq}} = -\frac{\mathcal{L}_f s}{\mathcal{L}_g s}, \quad u_{\mathrm{s}} = \frac{k}{\mathcal{L}_g s}\tanh(s) \end{aligned}$$

Where operator $\mathcal{L}$ stands for Lie derivative.

By selecting two surfaces the problem happening is illustrated.

$$\begin{aligned} s_1 &= 0.02x_1 + 0.43x_2 \\ s_2 &= 0.02x_1 + 0.43x_2 - \\ &- (2x_1^3 - 3x_1^2x_2 + 139x_1x_2^2 + 715.2x_2^3)10^{-4} \end{aligned}$$

State trajectories converge to sliding surface $s = s_1$ as it is depicted in Fig. 1 and will reach the origin. By adding nonlinear terms in sliding surface, an important problem is raised. Sliding surface (s_2) will roll in a boundary region near the origin as it is illustrated in Fig. 2. This phenomenon causes new equilibrium points (α and β) showing up in the phase plane, and the convergence of state trajectories to the origin is not assured anymore.

In the sliding phase the remained dynamics which have the order of $(n-m)$, not only are defined by sliding equation but also are restricted by system dynamics (The number of system states is denoted by n, and the number of inputs equals m). Consequently, here a systematic technique will be presented to overcome such difficulties. The main idea of our innovation is described as follows.

A stabilizing control signal $u_{\rm G}$ is designed for the nominal system. As it will be shown, there is a switching surface for which, by solving the equivalent control equation (3), $u_{\rm G}$ will be obtained and vice versa. Meaning that by solving $\dot{s}=0$, $u_{\rm G}$ exists and the sliding equation is computed by solving this equation for s. This design algorithm can give a stable sliding surface with higher order terms, which performs a better performance for the closed loop system in face of uncertainties or external disturbances. In addition an appropriate selection of the switching manifold extends the region of attraction, [10].

To investigate the stable property of sliding motion, first we proposed to design $u_{\rm eq}$ in a stable manner and then calculate the sliding equation. Assuming there is a $u_{\rm G}$ which globally stabilizes the nominal system (2). Dealing with this equation, it would be more convenient to define a $u_{\rm eq}$:

$$u_{\rm G} \;=\; u_{\rm eq} - a \tag{11}$$

Where in which a is a $m \times 1$ vector and denotes deviation of $u_{\rm G}$ from $u_{\rm eq}$.

$$u \;=\; u_{\rm G} - u_{\rm s}, \quad u_{\rm s} = k' \tanh(s) \tag{12}$$

By using a new definition of control law as given in (12), (7) will be reconstructed.

$$\begin{aligned} \dot{V} &= -s^{\rm T} k h(s) + s^{\rm T} U(\varpi - g a) \quad (13) \\ \dot{V} &< -s^{\rm T} k \,\mathrm{sign}(s) + s^{\rm T} U(\varpi - g a) \\ \dot{V} &< -\sum_{i=1}^{m} k_i |s_i| + \sum_{i=1}^{m} p_i s_i, p_i \triangleq \frac{\partial s_i}{\partial x}(\varpi - g a) \end{aligned}$$

By choosing a new gain matrix, the stability of reaching phase is accessible, i.e.,

$$k_i = |p_i| + \epsilon_i, \quad \epsilon_i > 0 \;\; \text{for} \;\; i = 1 \cdots m \tag{14}$$

The proposed control signal, instead of equivalent control law, causes some changes in the arrays of the gain matrix. In other words, it is possible to stabilize the sliding motion with a suitable control signal and eventually, the stability of the closed loop system is obtained by defining new values for k_i parameters. Changing these parameters to their new values will result in convergence of state trajectories toward the sliding manifold.

A smooth switch is used here, consequently $u_{\rm s} \simeq 0$, when the state trajectories reach surface. Then remaining part of control signal is $u_{\rm G}$, which assures the invariance of the sliding surface. If there is a form of $u_{\rm G}$ that stabilizes the sliding motion, then it is possible to establish stability conditions by using (7). The proposed approach, is step by step and can be followed systematically. First, the designer should find a control law, which stabilizes the nominal closed loop system globally. As previously denoted this system is not affected by external disturbances or any structural uncertainties and is described by (2). It must be reminded that, the computed control signal cannot globally stabilize system (1). To derive $u_{\rm G}$, one may use one of conventional control methods, such as adaptive, intelligent, robust, Lyapunov or other control based methods. To satisfy the stability condition in nonlinear systems, the most general form is to use Lyapunov based methods. In this regard, it is reasonable to use Global Stability Lemma (G.S.L.) as a suitable choice. For the purpose of developing the main results, this Lemma is reminded, [9].

Lemma 1: Consider a system described by equations of the form

$$\begin{aligned} \dot{z} &= f_{\rm z}(z, \xi) \\ \dot{\xi} &= u_{\xi} \end{aligned} \tag{15}$$

Suppose there exists a smooth real–valued function

$$\xi = v^{\star}(z) \ ,$$

with $v^{\star}(0) = 0$, and a smooth real–valued function $V(z)$, which is positive definite, such that

$$\frac{\partial V}{\partial z} f_{\rm z}(z, v^{\star}) < 0$$

for all nonzero z. Then, there exists a smooth static feedback law $u = u(z, \xi)$ with $u(0,0) = 0$, smooth real–valued function $W(z, \xi)$, which is positive definite, such that

$$\begin{pmatrix} \dfrac{\partial W}{\partial Z} & \dfrac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f_{\rm z}(z,\xi) \\ u(z,\xi) \end{pmatrix} < 0 \; \forall (z,\xi) \neq (0,0) \tag{16}$$

For proof see [9].

A systematic method for computing u_{ξ} has been presented in [9]. In this paper, it is intended to use the results of the Lemma 1 to modify the S.M.C. method. In this regard, the nominal system (2) should be converted in regular form. It is assumed that, there is a diffeomorphizm,

$$\begin{aligned} z &= [\phi_1(x) \;\cdots\; \phi_{n-m}] \\ \xi &= [\phi_{n-m+1}(x) \;\cdots\; \phi_n] \end{aligned}$$

which maps the nominal model into the regular form. For this transformation, the following conditions must be satisfied.

$$\det\Big(\frac{\partial \phi}{\partial x}\Big) \neq 0, \quad \frac{\partial \phi}{\partial x} g = [0_{m\times(n-m)} \;\; g_{\xi}]^{\rm T} \tag{17}$$

Transferred dynamics will be in regular form of (15). Considering $u_{\xi} = f_{\xi} + g_{\xi} u, g_{\xi} \in \mathbb{R}^{m\times m}$, the control signal is selected as $u = u_{\rm G} - u_{\rm s}$. Because of using the continuous approximation of the hard switch, when state trajectories reach the sliding surface, $u_{\rm s}$ will vanish. In this case, $u_{\rm G}$ should be designed such that the closed loop system is globally stable. From Lemma 1, u_{ξ} will be found and then $u_{\rm G}$ will be calculated as:

$$u_{\rm G} \;=\; g_{\xi}^{-1}(u_{\xi} - f_{\xi}(z,\xi)) \tag{18}$$

The control law defined by (18) will make the nominal system (2) stable. This approach is not robust, since the stability and performance of the closed loop system will be affected by external disturbances and model uncertainties. It should be noted that adding the switch part to this part forms a robust technique and has the ability to overcome the non–ideal conditions which affect the closed loop system. The switching part has a dominant role in the S.M.C. to overcome the system uncertainties and disturbances. To illuminate the lack of stability in sliding phase, we suggest to use G.S.L. for designing u_{G} instead of equivalent control signal. It is proved that this control signal will cause a stable sliding surface. Consequently, an appropriate method to design this surface which results in the stable sliding motion will be described in the next section.

IV. Optimal approach to compute the nonlinear stable surface

In this section optimal control theory is utilized to compute the non linear sliding surface which results in the optimal closed loop performance. To have a robust closed loop system,the modified sliding mode structure is considered. Now, consider the nominal system (2) with control law (12).

$$u = u_{\mathrm{G}}(x) - k' \tanh(s)$$

In this regard, sliding surface may be viewed as the input vector.

$$\begin{aligned} \dot{x} &= f(x) + g(x)\Big(u_{\mathrm{G}}(x) - k' \tanh(s)\Big) \\ \dot{x} &= \mathcal{F}(x) + \mathcal{G}(x)\tanh(\mathcal{U}), \quad \mathcal{U} = s(x) \end{aligned} \tag{19}$$

It is assumed that

$$\mathcal{F}(x) = f(x) + g(x)u_{\mathrm{G}}(x),\ \mathcal{G}(x) = -g(x)k',\ \mathcal{V} = \tanh(\mathcal{U}).$$

The optimal control problem is reminded here. First, a nonlinear system is assumed and performance index $G(x,\mathcal{V})$ is defined. This function may have a quadratic nonlinear form in general.

$$\dot{x} = \mathcal{F}(x) + \mathcal{G}(x)\mathcal{V}, \qquad x \in \mathbb{R}^n,\ \mathcal{V} \in \mathbb{R}^m \tag{20}$$

$$J(x,\mathcal{V}) = \int_0^\infty G(x,\mathcal{V})\mathrm{d}t \tag{21}$$

It is not possible to generally solve this optimization problem, however, there are some conditions which lead to the selection of a proper one.

Hamiltonian function is defined as:

$$H(x,\mathcal{V}) \triangleq G(x,\mathcal{V}) + \frac{\partial V(x)}{\partial x}\{\mathcal{F} + \mathcal{G}\mathcal{V}\}$$

The optimal control signal $\mathcal{V}$ may be computed by using Hamilton–Jacobi–Bellman equation. More details can be found in [11].

Theorem 1: Consider $V(x)$ as a Lyapunov function. The optimization problem

$$\min_{\mathcal{V}} J(x,\mathcal{V})$$

for the system (20), is equivalent to the following one:

$$\min_{\mathcal{V}} H(x,\mathcal{V})$$

Optimal response $\mathcal{V}_\star$ is obtained by solving the latter problem. In this manner $\mathcal{V}_\star$ will satisfy the following equations:

$$H(x,\mathcal{V}_\star(x,V_{\mathrm{x}}),V_{\mathrm{x}}) = 0, \quad V_{\mathrm{x}} \triangleq \frac{\partial V}{\partial x} \tag{22}$$

$$\frac{\partial H(x,\mathcal{V}_\star)}{\partial \mathcal{V}} = 0 \tag{23}$$

For proof see [11].

In order to derive the optimal response, a method based on Taylor series expansion is used. In this regard, one could expand $\mathcal{F} + \mathcal{G}\mathcal{V}$ and $G(x,\mathcal{V})$ as follows:

$$\begin{aligned} \mathcal{F} + \mathcal{G}\mathcal{V} &= Ax + B\mathcal{V} + \mathcal{F}_{\mathrm{r}}(x,\mathcal{V}) \\ G(x,\mathcal{V}) &= x^{\mathrm{T}}Qx + 2x^{\mathrm{T}}N\mathcal{V} + \mathcal{V}^{\mathrm{T}}R\mathcal{V} + G_{\mathrm{r}}(x,\mathcal{V}) \end{aligned}$$

Where $\mathcal{F}_{\mathrm{r}}$ and $\mathcal{G}_{\mathrm{r}}$ stand for higher order terms. As a necessary condition, matrices A, B, Q, N and R should be real and

$$\begin{pmatrix} Q & N \\ N^{\mathrm{T}} & R \end{pmatrix}$$

should be a positive definite matrix. Equations (22) is known as Hamilton–Jacobi–Bellman (HJB) equation. If mentioned conditions are satisfied, then it is possible to solve HJB equation and calculate $\mathcal{V}_\star$ in the form of Taylor expansion. Finally optimal nonlinear sliding surface $\mathcal{U}_\star$ is computed from $\mathcal{V} = \tanh(\mathcal{U})$.

By applying the nonlinear optimal theory to system described by (19) the optimal nonlinear surface is computed. The main parameters which affect the final response are N, R and Q. The level of direct energy consumption depends on these parameters. In this paper, a quadratic performance index is defined:

$$\begin{aligned} G(x,\mathcal{V}) &= x^{\mathrm{T}}Qx + \mathcal{V}^{\mathrm{T}}R\mathcal{V}, \quad \mathcal{V} = \tanh(\mathcal{U}) \\ \tanh(\mathcal{U}) &= [\tanh(\mathcal{U}_1)\cdots\tanh(\mathcal{U}_m)]^{\mathrm{T}} \\ & Q \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{m\times m} \end{aligned} \tag{24}$$

To reduce the energy consumption of the switching part, $\mathcal{V}^{\mathrm{T}}R\mathcal{V}$ is added to the $G(x,\mathcal{V})$. The main idea in optimal control is forming a desired performance for a closed loop system by which the transient and steady state behavior of state trajectories are affected by defining an appropriate function like $G(x,\mathcal{V})$. A certain selection of this function could improve the closed loop response. Another way to have a more precise response is to expand the number of terms in sliding equation. Direct results of this computation are increasing of domain of attraction, improving the response of the system and decreasing the level of consumed energy.

The main aspects of the proposed technique are summarized in the following theorem. To describe these items we present a case study in the next section.

Theorem 2: Consider a nominal system in the form of (2). Suppose there exists a diffeomorphizm through which,(2) may be transformed to the regular form of (15). Under the performance index given in (24), and having u_{G} from (18), the control input given in (12), the optimal sliding surface may be computed by solving the HJB equation of (22). The obtained nonlinear sliding surface results in a larger domain of attraction, and a lower level of energy consumption.

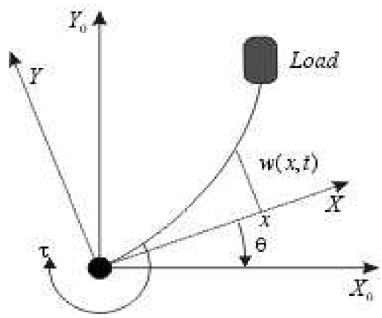

Fig. 3. Flexible–link structure

V. Case study: A flexible–link robot

Fig. 3 shows the structure of the flexible-link used. Assume that the flexible-link is uniformly elastic and is an Euler-Bernoulli beam with its torsion and strain set being zero. If the flexible-link doesn't have any deflection, it aligns on X axis. The deflection of any point of the link at any time is presented by $w(x,t)$ and is derived by solving Euler-Bernoulli beam equation.

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \quad (25)$$

The link cross–sectional area, Young's modulus, inertia of the link and its density are denoted by A, E, I and ρ, respectively. Mass per unit of length is denoted by $\gamma = \rho A$, and the Length of the link is L. By separating variables, the solution is obtained. The natural frequencies of the vibrations are denoted by ω_i. For measuring the deflection, a finite number of flexible modes are assumed (assumed modes method). This number depends on the flexibility of the manipulator and its vibration, [7]. By using recursive Lagrange approach, the dynamic equations are derived.

$$B(\theta,\delta)\begin{bmatrix}\ddot{\theta}\\ \ddot{\delta}\end{bmatrix} + \begin{bmatrix}n_1(\dot{\theta},\delta,\dot{\delta}) + f_c + f_v + d\\ n_2(\theta,\dot{\delta}) + K_s\,\delta + F_s\,\dot{\delta}\end{bmatrix} = \begin{bmatrix}\tau\\ 0\end{bmatrix}$$
$$\theta_t = \theta + \Phi_e^T\delta$$

θ is the hub angle, θ_t defines the end point position, δ is an $\ell \times 1$ vector of deflection variables which ℓ represents the number of assumed modes, n_1 and n_2 demand the coriolis and centrifuge forces and B is a positive definite symmetric matrix which is known as inertia matrix. $K_s = \mathrm{Diag}\{k_{si} \ \cdots \ k_{s\ell}\}$ is the stiffness coefficient and it is a positive definite diagonal matrix. f_v denotes the viscous friction in the hub position, the structural damping is shown by $F_s = \mathrm{Diag}\{f_{s1} \ \cdots \ f_{s\ell}\}$ that is a positive definite diagonal matrix which represents the internal viscous friction of the flexible structure and f_c is the Coulomb friction usually modeled by a sign function. However, to eliminate discontinuity it is modeled by:

$$\begin{aligned} f_c &= C_c\tanh(\alpha\dot{\theta}), \qquad \alpha, C_c > 0 \quad (26)\\ f_v &= C_v\dot{\theta}\end{aligned}$$

TABLE I
Link parameters for the experimental manipulator

C_v	0.59	Nm/rads^{-1}
C_c	4.72 for $\dot{\theta} > 0$, 4.77 for $\dot{\theta} \le 0$	Nm
EI	2.21	Nm^2
ω_1, ω_2	3.6, 16.73	rad/s
m_b	1.356	Kg
M_L	0.05	Kg
L	1.2	m
$\gamma = \rho A$	1.13	Kg/m
J_b	0.651	Kgm^2
I_0	0.27	Kgm^2
J_L	0.0116	Kgm^2

Table I shows the parameters of a real link where, I_0 denotes the joint actuator inertia and J_b is the link inertia relative to the joint, C_v represents the hub damping coefficient, C_c is the coulomb friction coefficient. M_L is the mass of the payload. In this case we have considered two vibration modes ($\ell = 2$). The link mass illustrated by m_b and the load inertia is given by J_L, [7], [8].

By applying the following transformation, the FLR dynamics will be converted in regular form.

$$\begin{aligned}\begin{bmatrix}\hat{\theta}_2\\ \hat{\delta}_2\end{bmatrix} &= B(\theta_1,\delta_1)\begin{bmatrix}\theta_2\\ \delta_2\end{bmatrix}, \quad \theta_1 = \theta, \delta_1 = \delta, \theta_2 = \dot{\theta}, \delta_2 = \dot{\delta}\\ z &= \begin{bmatrix}\theta_1^T & \delta_1^T & \hat{\delta}_2^T\end{bmatrix}^T\\ \xi &= \hat{\theta}_2\end{aligned}$$

The final form of system equations is like (15).

$$\begin{cases}\dot{z} = f_z(z,\xi)\\ \dot{\xi} = u_\xi, \qquad u_\xi = f_\xi(z,\xi) + \tau\end{cases}$$

By using the proposed method, the sliding surface is obtained. It is possible to calculate higher order terms of sliding manifold. Here, the first, second and third order terms of sliding surface are calculated, while the second term is calculated to equal zero. $s^{[i]}$ denotes terms of i degree.

$$\begin{aligned} s^{[1]} &= 1.335\,z_1 - 2.773\,z_2 - 0.059\,z_3 + 1.619\,\xi\\ s^{[2]} &\simeq 0\\ s^{[3]} &= 0.793\,{z_1}^3 - 4.989\,{z_1}^2 z_2 - 0.108\,{z_1}^2 z_3 +\\ &\quad + 10.051\,z_1{z_2}^2 + 0.27\,z_1z_2z_3 - 11.862\,z_1z_2\xi -\\ &\quad - 0.244\,z_1z_3\xi + 3.504\,z_1\xi^2 - 7.736\,{z_2}^3 +\\ &\quad + 11.536\,{z_2}^2\xi + 0.053\,z_2{z_3}^2 + 0.156\,z_2z_3\xi +\\ &\quad + 0.005\,{z_3}^3 - 0.0227\,{z_3}^2\xi - 0.11\,z_3\xi^2 +\\ &\quad + 2.894\,{z_1}^2\xi - 0.002\,z_1{z_3}^2 + 0.224\,{z_2}^2 z_3 +\\ &\quad + 1.396\,\xi^3 - 6.922\,z_2\xi^2\\ s_L &= s^{[1]}\\ s_N &= s^{[1]} + s^{[2]} + s^{[3]}\end{aligned}$$

Linear (s_L) and nonlinear (s_N) surfaces are applied to the flexible–link robot model without considering external disturbances and structural uncertainties. In this case, the nonlinear sliding equation has provided a better performance.

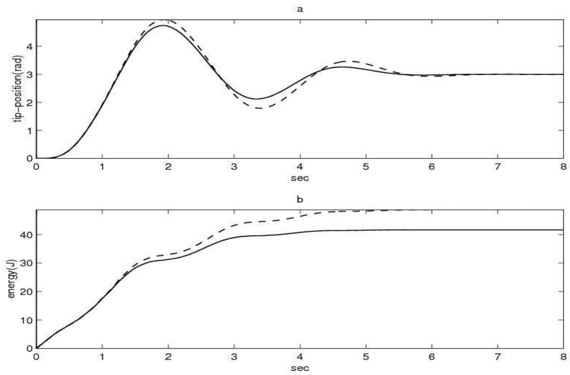

Fig. 4. Response of flexible robot for linear (dashed), and nonlinear (solid) surfaces with structural uncertainty; (a) tip–position and (b) index function.

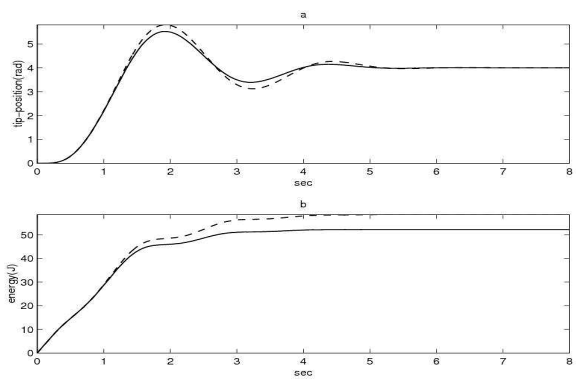

Fig. 5. Response of flexible robot for linear (dashed), and nonlinear (solid) surfaces with disturbance equals $\sin(4t)$; (a) tip–position and (b) index function.

Effectiveness of this method has been verified through simulation results.

Structural uncertainties may happen in system dynamics. In the flexible–link robot model, parameters like friction coefficient and characteristics of payload are not exactly known. It is assumed that these parameters experience deviations from their nominal values. Simulation results show a better performance of nonlinear sliding surface (See Fig. 4). External disturbances may affect any system. Here, it is assumed that a sinusoidal signal like $\sin(4t)$ influences the closed loop system at the input channel. The corresponding simulation results are presented in Fig. 5. As a basic result, it must be noted that a nonlinear surface will result in a larger domain of attraction, [10]. Fig. 6 shows clearly the extension of domain of attraction in case of using a nonlinear sliding equation. Note that in this case, the set point value is larger with respect to those of previous figures.

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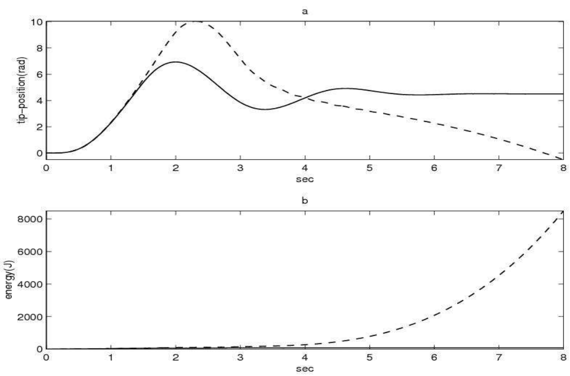

Fig. 6. Comparison of domain of attraction for linear (dashed), and nonlinear (solid) surfaces without uncertainty; (a) tip–position and (b) index function.

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