Delay Compensated Control of the Stefan Problem

Shumon Koga, Miroslav Krstic

University of California, San Diego

CDC 2017
Motivation

Stefan Problem (Phase Change Model)

3D-Printing

Lithium Ion Batteries

Cryosurgery

Sea Ice
Physical Model: Melting

![Diagram of melting process]

- Liquid region
- Solid region
- Heat flux: \( q_c(t) \)
- Temperature: \( T(x, t) \)
- Melting point: \( T_m \)
- Distance: \( s(t) \)
- Length: \( L \)
Physical Model: Melting + Actuator Delay
During the process

**Objective:** Design heat control $q_c(t)$ to achieve

$$s(t) \to s_r, \quad T(x,t) \to T_m, \quad \text{as} \quad t \to \infty$$
The Stefan Problem theoretically. Although the Stefan Problem estimation error and output feedback systems of one-phase stability. The main contribution of this paper is that, boundary output feedback controller that achieves the exponential stability of sum of the moving interface.

Along this paper we proposed an observer design and this is the first result which shows the convergence of some initial conditions, which guarantees some physical future work.
The main contribution of this paper is that, some initial conditions, which guarantees some physical boundary output feedback controller that achieves the exponential stability of sum of the moving interface.

A nonlinear backstepping transformation for moving boundary problem is utilized and the controller is proved to keep positive with a measurement of the moving interface.

Along this paper we proposed an observer design and it is investigated as a future work.


Panagiotis D. Christofides. Robust control of parabolic PDE systems. Siam, 2008.


Shuxia Tang and Chengkang Xie. Stabilization for a coupled PDE-ODE system, with an application to the liquid solid delay.


C. B. Peschanski and J. C. Kantor. Backstepping control of the two-phase Stefan problem, with an application to the liquid solid delay.


C. B. Peschanski and J. C. Kantor. Backstepping control of the two-phase Stefan problem, with an application to the liquid solid delay.


Panagiotis D. Christofides. Robust control of parabolic PDE systems. Siam, 2008.


Shuxia Tang and Chengkang Xie. Stabilization for a coupled PDE-ODE system, with an application to the liquid solid delay.


C. B. Peschanski and J. C. Kantor. Backstepping control of the two-phase Stefan problem, with an application to the liquid solid delay.


Panagiotis D. Christofides. Robust control of parabolic PDE systems. Siam, 2008.


Shuxia Tang and Chengkang Xie. Stabilization for a coupled PDE-ODE system, with an application to the liquid solid delay.


C. B. Peschanski and J. C. Kantor. Backstepping control of the two-phase Stefan problem, with an application to the liquid solid delay.


Panagiotis D. Christofides. Robust control of parabolic PDE systems. Siam, 2008.


Shuxia Tang and Chengkang Xie. Stabilization for a coupled PDE-ODE system, with an application to the liquid solid delay.


Stefan Problem theoretically. Although the Stefan Problem some initial conditions, which guarantees some physical boundary output feedback controller that achieves the of stability. The main contribution of this paper is that, \( s(t) \), state \( \tilde{t} \), \( 0.36 \), \( \epsilon \), \( 0.002 \), \( 0.006 \), \( 0.002 \), \( 0.31 \), \( 0.32 \), \( 0.36 \), \( \epsilon \), \( 0.32 \), \( 0.37 – 0.04 \), \( 0.04 \), \( t \), \( 20 \), \( 80 \), \( 100 \), \( \text{OutputFB} \), \( \text{StateFB} \), \( 0.36 \), \( \text{Time} (\text{s}) \), \( \text{Future} \).

**Fig. 1.** The moving interface.

**Fig. 3.** The positiveness verification of the controller.

**PDE**

\[
T_t(x, t) = \alpha T_{xx}(x, t), \quad 0 < x < s(t) < L \\
-kT_x(0, t) = q_c(t - D) \\
T(s(t), t) = T_m
\]
The main contribution of this paper is that, along this paper we proposed an observer design and estimation error and output feedback systems of one-phase backstepping transformation for moving boundary problem is utilized and the controller is proved to keep positive with exponential stability of sum of the moving interface, boundary output feedback controller that achieves the $\|\cdot\|_2$-norm of the temperature, and estimation error of them.

$$T_t(x, t) = \alpha T_{xx}(x, t), \quad 0 < x < s(t) < L$$

$$-k T_x(0, t) = q_c(t-D)$$

$$T(s(t), t) = T_m$$

$$\dot{s}(t) = -\beta T_x(s(t), t)$$
PDE \[ T_t(x, t) = \alpha T_{xx}(x, t), \quad 0 < x < s(t) < L \]
\[-kT_x(0, t) = q_c(t-D)\]
\[ T(s(t), t) = T_m \]

ODE \[ \dot{s}(t) = -\beta T_x(s(t), t) \]

State-dependent moving boundary \(\rightarrow\) Nonlinear
**Assumption**: Initial interface position $s_0 > 0$, and initial temperature $T_0(x)$ is Lipschitz ($H \equiv \text{Lip. const.}$)

$$0 < T_0(x) - T_m < H(s_0 - x)$$

**Assumption**: The past input maintains non-negative, i.e.

$$q_c(t) \geq 0, \quad -D < \forall t < 0.$$
Model valid iff

\[ T(x, t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0 \]

How to guarantee this?
Model valid iff

\[ T(x, t) > T_m, \quad \text{for } \forall x \in (0, s(t)), \quad \forall t > 0 \]

How to guarantee this?

**Lemma** If \( q_c(t) > 0 \) \( \forall t > 0 \), then \( \dot{s}(t) > 0 \) \( \forall t > 0 \) and

\[ T(x, t) > T_m, \quad \forall x \in (0, s(t)), \forall t > 0 \]
Energy Conservation

\[
\frac{d}{dt} \left( \frac{k}{\alpha} \int_0^{s(t)} (T(x, t) - T_m) \, dx + \frac{k}{\beta} s(t) + \int_{t-D}^{t} q_c(\theta) \, d\theta \right) = q_c(t) > 0
\]
Energy Conservation

\[
\frac{d}{dt} \left( \frac{k}{\alpha} \int_0^{s(t)} (T(x, t) - T_m) dx + \frac{k}{\beta} s(t) + \int_{t-D}^{t} q_c(\theta) d\theta \right) = q_c(t) > 0
\]

- Internal Energy
- Stored Energy
- Work

For model to be valid (single melting interface), heat must be added.
Energy Conservation

\[ \frac{d}{dt} \left( \frac{k}{\alpha} \int_0^{s(t)} (T(x, t) - T_m) dx + \frac{k}{\beta} s(t) + \int_{t-D}^{t} q_c(\theta) d\theta \right) = q_c(t) > 0 \]

- For model to be valid (single melting interface), heat must be added.

- When heat added, total energy (internal + stored) grows.
Energy Conservation

\[
\frac{d}{dt} \left( \frac{k}{\alpha} \int_0^{s(t)} (T(x, t) - T_m) dx + \frac{k}{\beta} s(t) + \int_{t-D}^{t} q_c(\theta) d\theta \right) = q_c(t) > 0
\]

- For model to be valid (single melting interface), heat must be added.

- When heat added, total energy (internal + stored) grows.

- Since total energy grows, energy corresponding to setpoint must be greater than initial energy.
The following assumption necessary (because $\int_0^{s_r} (T_r(x) - T_m) dx = 0$)

**Assumption**: Setpoint $s_r$ chosen to satisfy

$$s_r > s_0 + \beta \left( \frac{1}{\alpha} \int_0^{s_0} (T_0(x) - T_m) dx + \int_0^0 \frac{q_c(t)}{k} dt \right)$$
The control law

\[ q_c(t) = -c \left( \int_{t-D}^{t} q_c(\theta) d\theta + \frac{k}{\alpha} \int_{0}^{s(t)} (T(x, t) - T_m) dx + \frac{k}{\beta} (s(t) - s_r) \right), \]

where \( c > 0 \), makes the closed-loop system **globally exponentially stable** in the norm

\[ \|T - T_m\|_{\mathcal{H}_1}^2 + (s - s_r)^2. \]
**Theorem**  The control law

\[ q_c(t) = -c \left( \int_{t-D}^{t} q_c(\theta) d\theta + \frac{k}{\alpha} \int_{0}^{s(t)} (T(x, t) - T_m) dx + \frac{k}{\beta} (s(t) - s_r) \right), \]

where \( c > 0 \), makes the closed-loop system globally exponentially stable in the norm

\[ ||T - T_m||_{\mathcal{H}_1}^2 + (s - s_r)^2. \]

**Note**: Control law is nonlinear because of \( s(t) \) in integration limit.
Explanation of Design

Reference errors

\[ u(x, t) := T(x, t) - T_m, \quad X(t) := s(t) - s_r \]

Change of variable

\[ v(x, t) := q_c(t - x - D)/k \]
Explaination of Design

Reference errors

\[ u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r \]

Change of variable

\[ v(x,t) := q_c(t - x - D)/k \]

\((v, u, X)\)-system

\[ v_t(x,t) = -v_x(x,t), \quad -D < x < 0 \]
\[ v(-D,t) = q_c(t), \]
\[ u_x(0,t) = -v(0,t), \]
\[ u_t(x,t) = \alpha u_{xx}(x,t), \quad 0 < x < s(t) \]
\[ u(s(t),t) = 0, \]
\[ \dot{X}(t) = -\beta u_x(s(t),t). \]

\[ \text{PDE} \]
\[ v_t = -v_x, \]

\[ \text{PDE} \]
\[ u_t = \alpha u_{xx}, \]
\[ u(s(t),t) = 0, \]

\[ \text{ODE} \]
\[ \dot{X}(t) = -\beta u_x(s(t),t) \]
Explanation of Design

Backstepping transformations

\[
\begin{align*}
w(x, t) &= u(x, t) - \frac{c}{\alpha} \int_x s(t) (x - y)u(y, t) dy - \frac{c}{\beta} (x - s(t))X(t) \\
z(x, t) &= v(x, t) + c \int_x^0 v(y, t) dy + \frac{c}{\alpha} \int_0 s(t) u(y, t) dy + \frac{c}{\beta} X(t)
\end{align*}
\]

\(\rightarrow\) both are \textit{nonlinear} transformations
Explanation of Design

Backstepping transformations

\[ w(x, t) = u(x, t) - \frac{c}{\alpha} \int_x^{s(t)} (x - y)u(y, t) \, dy - \frac{c}{\beta}(x - s(t))X(t) \]

\[ z(x, t) = v(x, t) + c \int_0^x v(y, t) \, dy + \frac{c}{\alpha} \int_0^{s(t)} u(y, t) \, dy + \frac{c}{\beta}X(t) \]

→ both are nonlinear transformations

Target system

\[ z_t(x, t) = -z_x(x, t), \quad -D < x < 0 \]

\[ z(-D, t) = 0, \]

\[ w_x(0, t) = -z(0, t), \]

\[ w_t(x, t) = \alpha w_{xx}(x, t) + \frac{c}{\beta} \dot{s}(t)X(t), \quad 0 < x < s(t) \]

\[ w(s(t), t) = 0, \]

\[ \dot{X}(t) = -cX(t) - \beta w_x(s(t), t). \]
Model validity

**Proposition**  Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$. 
Model validity

**Proposition**  Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

Proof:

$$q_c = -c \text{ (total energy} - \text{setpoint energy)}$$

$$\dot{q}_c(t) = -cq_c(t) \quad \therefore q_c(t) = q_c(0)e^{-ct} > 0$$
Model validity

**Proposition**  Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$.

Proof:

$$q_c = -c (\text{total energy} - \text{setpoint energy})$$

$$\dot{q}_c(t) = -cq_c(t) \quad \therefore q_c(t) = q_c(0)e^{-ct} > 0$$

Lyapunov analysis with $\dot{s}(t) > 0$ and $s_0 < s(t) < s_r$ yields the norm estimate

$$\psi(t) \leq M \psi(0)e^{-bt},$$

for some positive constants $M > 0$ and $b > 0$, where

$$\psi(t) = \|v\|_{\mathcal{H}_1(-D,0)}^2 + \|u\|_{\mathcal{H}_1(0,s(t))}^2 + X(t)^2.$$
Numerical Simulation

Zinc

No overshoot
Numerical Simulation

Compare with uncompensated control

\[ q_c(t) = -c \left( \frac{k}{\alpha} \int_0^s(t) (T(x, t) - T_m) \, dx + \frac{k}{\beta} (s(t) - s_r) \right), \]

Uncompensated control violates the model validity
Numerical Simulation

Heat input

Temp. at $x=0$

Uncompensated control violates the model validity
Future Work

- Observer design under sensor delay
- Adaptive control under unknown delay