Output Feedback Control of the One-Phase Stefan Problem

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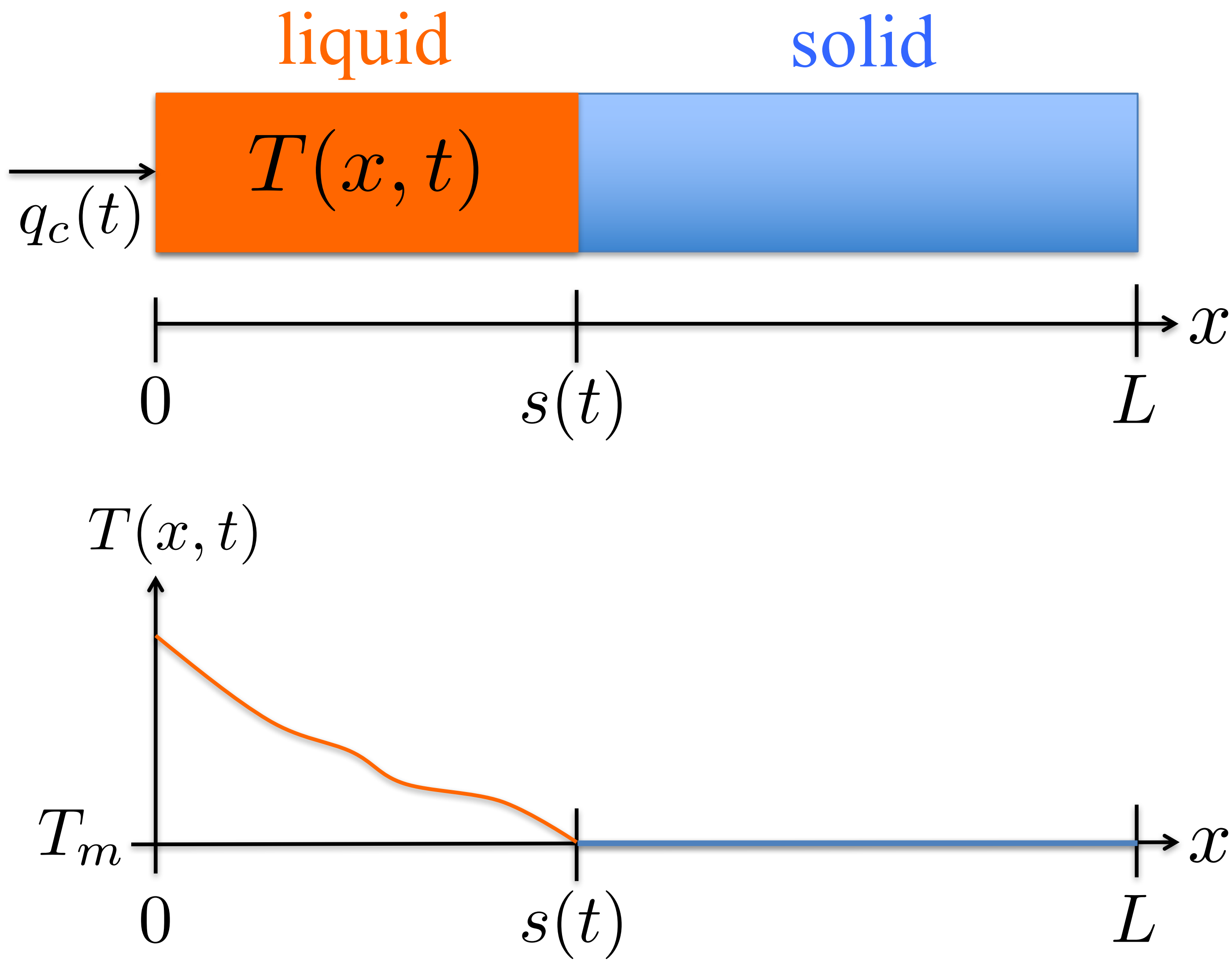
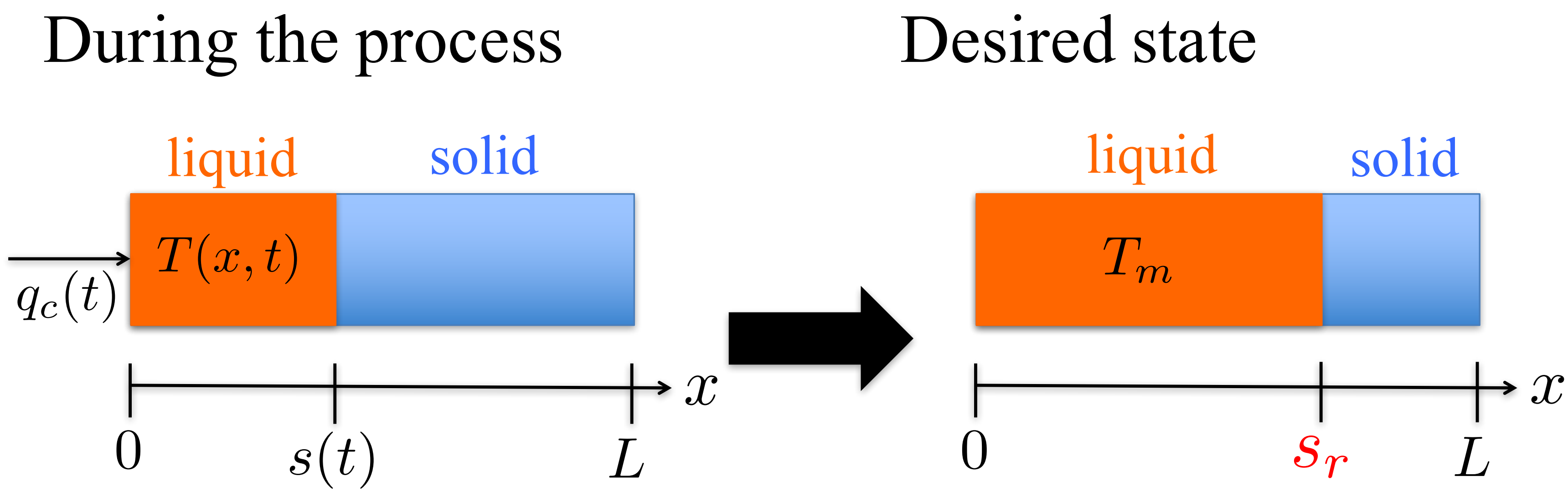
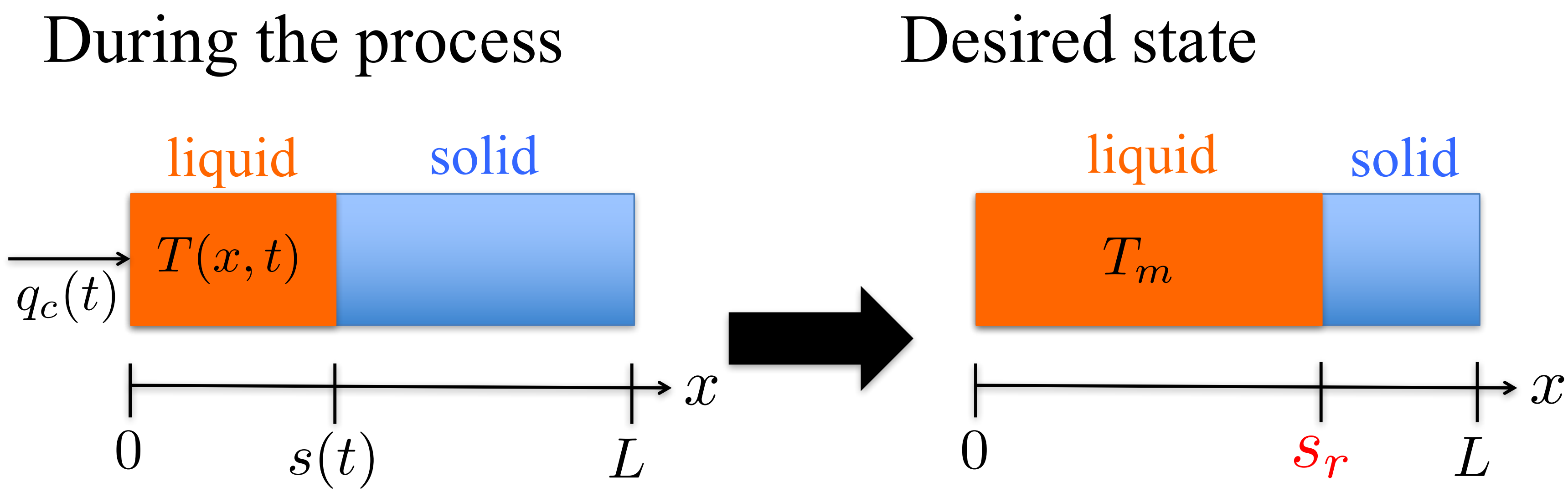
Outline

- Problem Statement
- Observer Design
- Output Feedback Control
- Future Works

Problem Statement


liquid
solid
$T(x,t)$
$q_c(t)$
0
$s(t)$
L
x
$T(x,t)$
T_m
0
$s(t)$
L
x

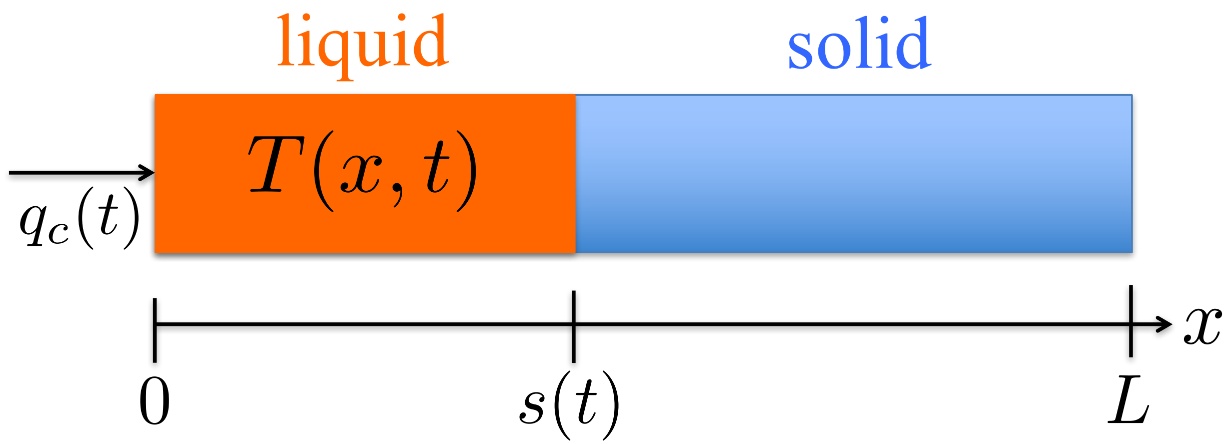

During the process
liquid
solid
$T(x,t)$
$q_c(t)$
x
0
$s(t)$
L
Desired state
liquid
solid
T_m
x
0
s_r
L



Objective: Design heat control $q_c(t) > 0$ to achieve

$$\boxed{s(t) \to s_r,} \quad \boxed{T(x,t) \to T_m,} \quad \text{as} \quad t \to \infty$$

with measurement of $s(t)$.


liquid
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x
0
$s(t)$
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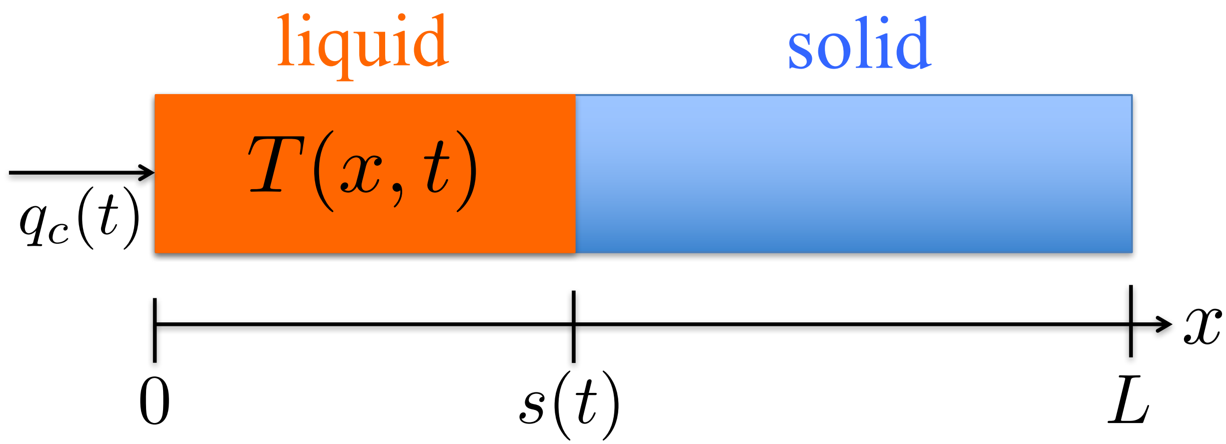
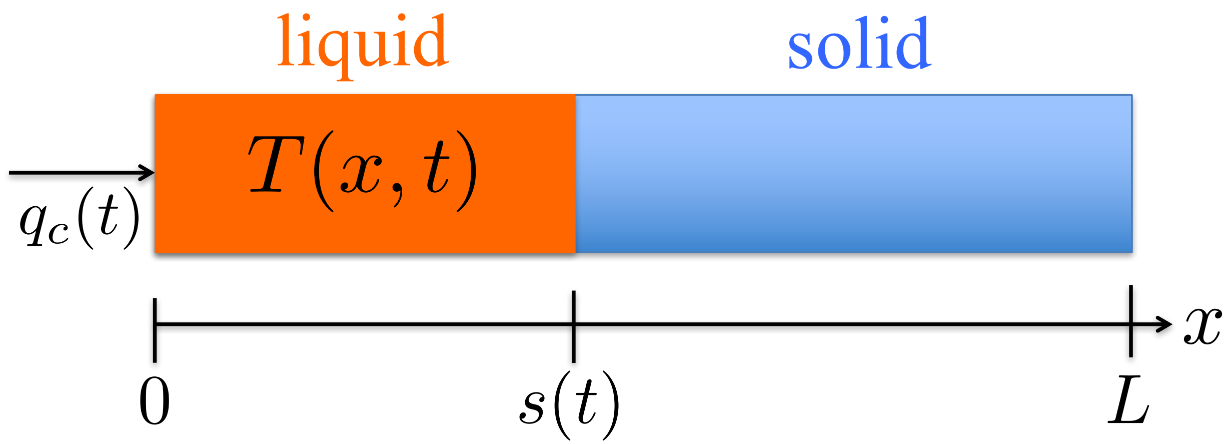


PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$



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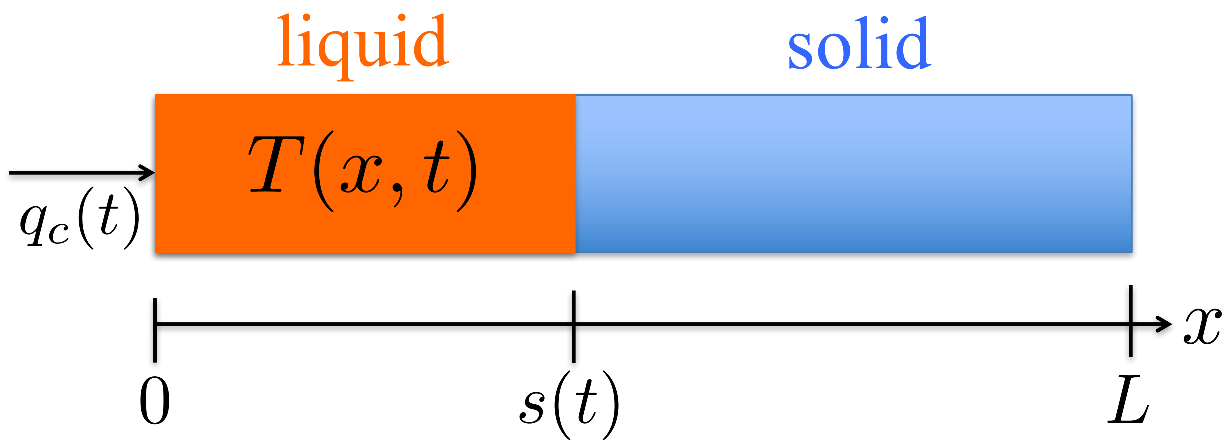
ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t)$



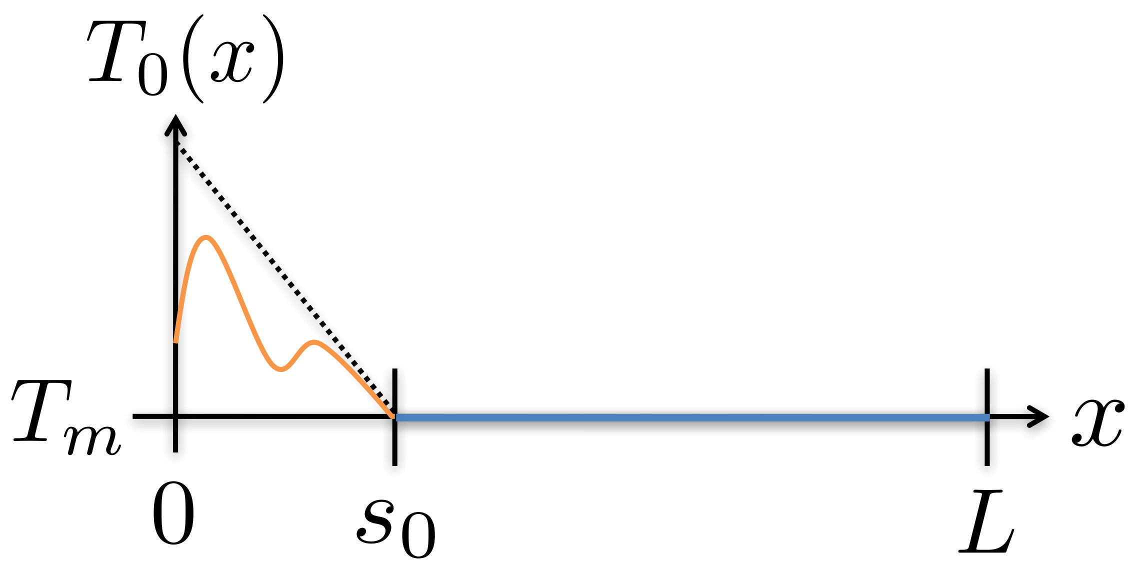
$$\text{PDE} \qquad \begin{aligned} T_t(x,t) &= \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L \\ T_x(0,t) &= -q_c(t)/k \\ T(s(t),t) &= T_m \end{aligned}$$

$$\text{ODE} \qquad \dot{s}(t) = -\beta T_x(s(t),t)$$

State-dependent moving boundary $\rightarrow$ Nonlinear

Assumption : Initial interface position $s_0 > 0$, and initial temperature $T_0(x)$ is Lipschitz ($H :=$ Lip. const.)

$$0 < T_0(x) - T_m < H(s_0 - x)$$



Model valid iff

$$\boxed{T(x,t) > T_m,} \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

How to guarantee this?

Model valid iff

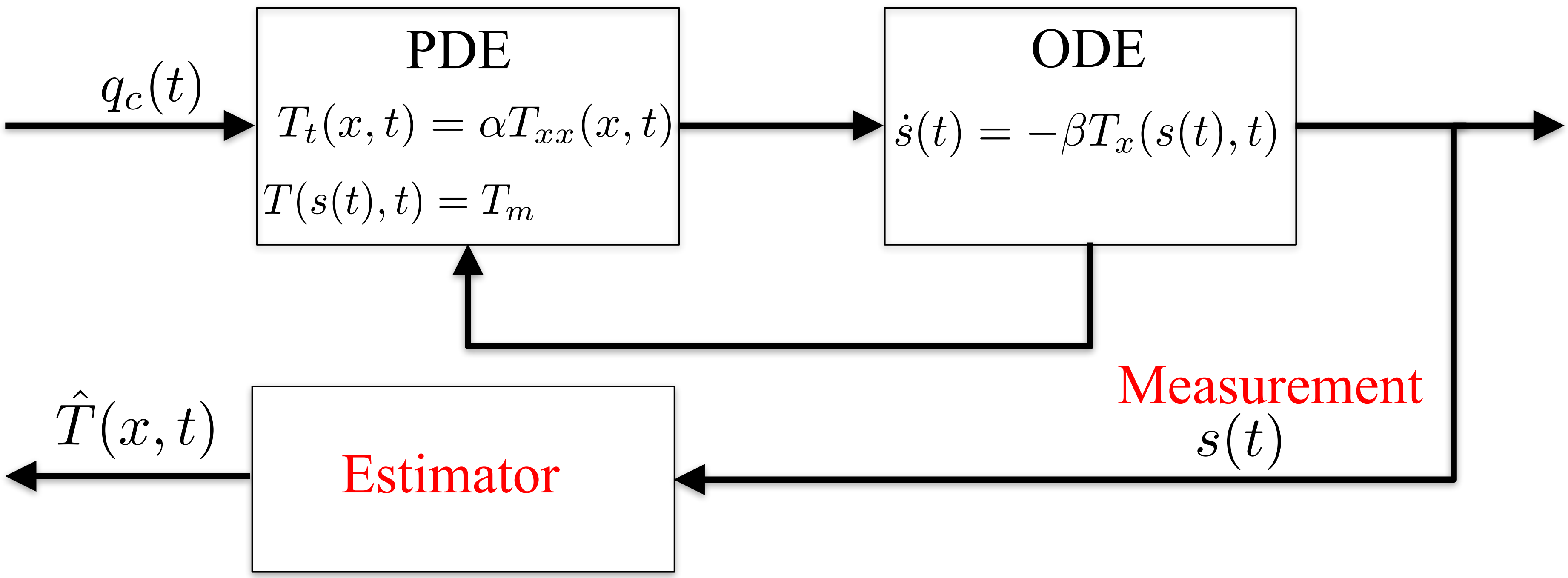
$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$

How to guarantee this?

Lemma If $\boxed{q_c(t) > 0}$ $\forall t > 0$, then $\dot{s}(t) > 0$ $\forall t > 0$ and

$$T(x,t) > T_m, \; \forall x \in (0, s(t)), \forall t > 0$$

Observer Design


PDE
$T_t(x,t) = \alpha T_{xx}(x,t)$
$T(s(t),t) = T_m$
ODE
$\dot{s}(t) = -\beta T_x(s(t),t)$
$q_c(t)$
Measurement
$s(t)$
Estimator
$\hat{T}(x,t)$

Theorem The observer design

$$\begin{aligned}\hat{T}_t(x,t) =& \alpha \hat{T}_{xx}(x,t) - P_1(x,s(t))\left(\frac{\dot{s}(t)}{\beta} + \hat{T}_x(s(t),t)\right), \\ -k\hat{T}_x(0,t) =& q_c(t), \\ \hat{T}(s(t),t) =& T_m,\end{aligned}$$

with the observer gain

$$P_1(x,s(t)) = -\lambda s(t) \frac{I_1\left(\sqrt{\frac{\lambda}{\alpha}\left(s(t)^2 - x^2\right)}\right)}{\sqrt{\frac{\lambda}{\alpha}\left(s(t)^2 - x^2\right)}}$$

where $\lambda > 0$ is a free parameter, makes the closed-loop system globally exponentially stable in the norm

$$||T - \hat{T}||^2_{\mathcal{H}_1}.$$

if $\dot{s}(t) > 0$.

Explanation of Design

Error Dynamics $(\tilde{u}(x,t) := T(x,t) - \hat{T}(x,t))$

$$\begin{aligned}\tilde{u}_t(x,t) =& \alpha\tilde{u}_{xx}(x,t) - P_1(x,s(t))\tilde{u}_x(s(t),t),\\ \tilde{u}(s(t),t) =& 0, \quad \tilde{u}_x(0,t) = 0\end{aligned}$$

Explanation of Design

Error Dynamics ($\tilde{u}(x,t) := T(x,t) - \hat{T}(x,t)$)

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\begin{aligned}
\tilde{u}_t(x,t) =& \alpha \tilde{u}_{xx}(x,t) - P_1(x, s(t)) \tilde{u}_x(s(t), t), \\
\tilde{u}(s(t), t) =& 0, \quad \tilde{u}_x(0,t) = 0
\end{aligned}
$$

Backstepping transformation

$$
\begin{aligned}
\tilde{u}(x,t) =& \tilde{w}(x,t) + \int_x^{s(t)} P(x,y) \tilde{w}(y,t) dy, \\
\tilde{w}(x,t) =& \tilde{u}(x,t) - \int_x^{s(t)} Q(x,y) \tilde{u}(y,t) dy,
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Backstepping transformation

$$\begin{aligned} \tilde{u}(x,t) =& \tilde{w}(x,t) + \int_x^{s(t)} P(x,y)\tilde{w}(y,t)dy, \\ \tilde{w}(x,t) =& \tilde{u}(x,t) - \int_x^{s(t)} Q(x,y)\tilde{u}(y,t)dy, \end{aligned}$$

Target system

$$\begin{aligned} \tilde{w}_t(x,t) =& \alpha \tilde{w}_{xx}(x,t) - \lambda \tilde{w}(x,t), \\ \tilde{w}(s(t),t) =& 0, \quad \tilde{w}_x(0,t) = 0 \end{aligned}$$

stable in $\mathcal{L}_2$ norm, and stable in $\mathcal{H}_1$ norm if $\dot{s}(t) > 0$.

Explanation of Design

The explicit solution of the gain kernel

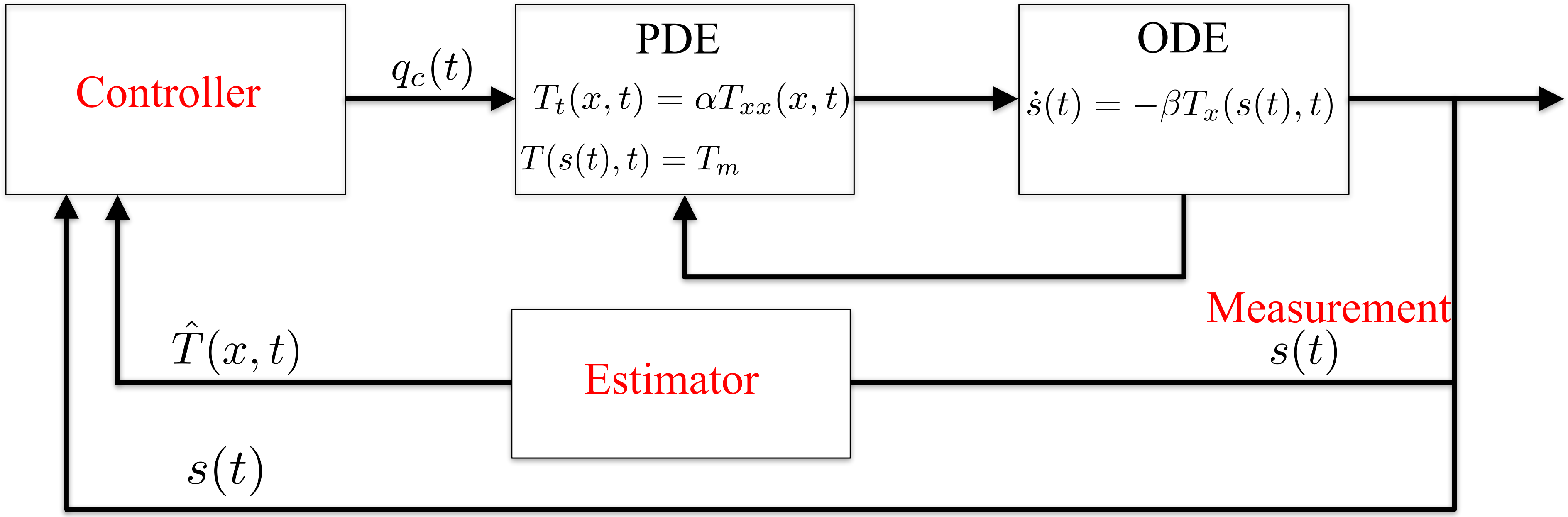
$$P(x,y) = \frac{\lambda}{\alpha} y \frac{I_1\left(\sqrt{\frac{\lambda}{\alpha}(y^2-x^2)}\right)}{\sqrt{\frac{\lambda}{\alpha}(y^2-x^2)}}$$

$I_1()$: modified Bessel function of 1st kind.

The observer gain must satisfy

$$\begin{aligned} P_1(x,s(t)) &= -\alpha P(x,s(t)) \\ &= -\lambda s(t) \frac{I_1\left(\sqrt{\frac{\lambda}{\alpha}\left(s(t)^2-x^2\right)}\right)}{\sqrt{\frac{\lambda}{\alpha}\left(s(t)^2-x^2\right)}} \end{aligned}$$

Output Feedback Control


Controller
$q_c(t)$
PDE
$T_t(x,t) = \alpha T_{xx}(x,t)$
$T(s(t),t) = T_m$
ODE
$\dot{s}(t) = -\beta T_x(s(t),t)$
$\hat{T}(x,t)$
Estimator
Measurement
$s(t)$
$s(t)$

Theorem The designed observer and the associated output feedback control law

$$q_c(t) = -ck\left(\frac{1}{\alpha}\int_0^{s(t)}\left(\hat{T}(x,t)-T_m\right)dx+\frac{1}{\beta}\left(s(t)-s_r\right)\right)$$

with a choice of

$$T_m+\hat{H}_l(s_0-x)\leq \hat{T}_0(x)\leq T_m+\hat{H}_u(s_0-x),$$

$$\lambda < \frac{4\alpha}{s_0^2}\frac{\hat{H}_l-H}{\hat{H}_u},$$

$$s_r > s_0+\frac{\beta s_0^2}{2\alpha}\hat{H}_u,$$

where $\hat{H}_u \geq \hat{H}_l > H$, makes the closed-loop system globally exponentially stable in

$$||T-\hat{T}||^2_{\mathcal{H}_1}+||T-T_m||^2_{\mathcal{H}_1}+(s(t)-s_r)^2.$$

Explanation of Design

Reference errors

$$\widehat{u}(x,t) := \widehat{T}(x,t) - T_m, \quad X(t) := s(t) - s_r$$

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$$\widehat{w}(x,t) = \widehat{u}(x,t) - \frac{c}{\alpha}\int_x^{s(t)} (x-y)\widehat{u}(y,t)dy - \frac{c}{\beta}(x - s(t))X(t),$$

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Target System

$$\begin{aligned}
\hat{w}_t(x,t) =& \alpha\hat{w}_{xx}(x,t) + \frac{c}{\beta}\dot{s}(t)X(t) + f(x, s(t))\tilde{w}_x(s(t),t),\\
\hat{w}(s(t),t) =& 0, \quad \hat{w}_x(0,t) = 0,\\
\dot{X}(t) =& -cX(t) - \beta\hat{w}_x(s(t),t) - \beta\tilde{w}_x(s(t),t)
\end{aligned}$$

Model validity

Lemma $\tilde{u}(x,t) < 0$ & $\tilde{u}_x(s(t),t) > 0$
if $\widehat{T}_0(x)$ & λ satisfy the given inequalities.

Proof is by maximum principle

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Proof is by maximum principle

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$
if s_r satisfies the given inequality.

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Lemma $\tilde{u}(x,t) < 0$ & $\tilde{u}_x(s(t),t) > 0$
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Proof is by maximum principle

Proposition Controller maintains $q_c(t) > 0$ and $s_0 < s(t) < s_r$
if s_r satisfies the given inequality.

Proof:

$$\begin{aligned}
\dot{q}_c(t) =& -cq_c(t) + \left(1 + \int_0^{s(t)} P(x, s(t))dx\right) \tilde{u}_x(s(t),t) \\
\geq& -cq_c(t) \\
\therefore q_c(t) \geq& q_c(0)e^{-ct} > 0
\end{aligned}$$

Lyapunov analysis

$$V := ||\widehat{w}||^2_{\mathcal{H}_1} + pX^2 + d||\tilde{w}||^2_{\mathcal{H}_1}$$

$$\begin{aligned}\dot{V} &\leq -bV + \dot{s}(t)\left(m_1 X(t)||\widehat{w}||_{\mathcal{L}_2} - m_2 \widehat{w}_x(s(t),t)^2\right)\\ &\leq -bV + a\dot{s}(t)V, \quad \because \dot{s}(t) > 0\end{aligned}$$

Overall Lyapunov functional

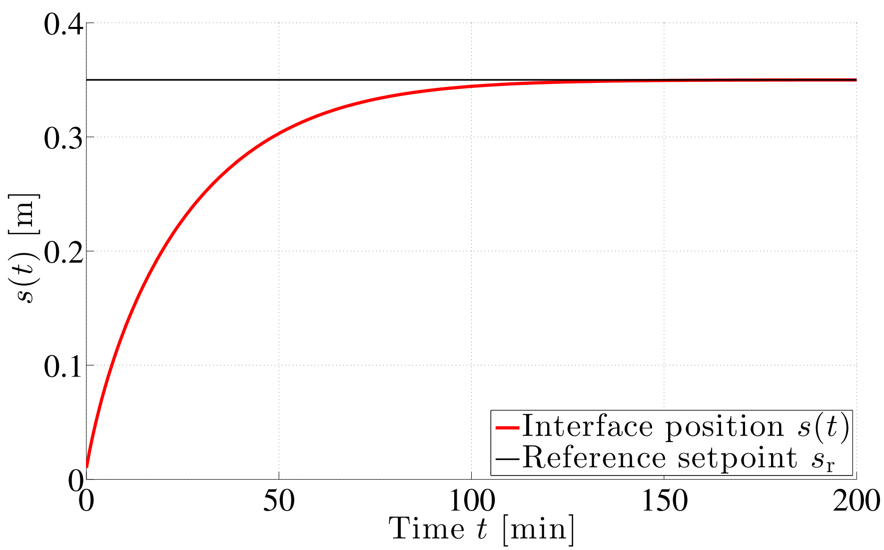
$$W := \frac{V}{e^{as}} = \frac{||\widehat{w}(T,s)||^2_{\mathcal{H}_1} + p(s-s_r)^2 + d||\tilde{w}(T,s)||^2_{\mathcal{H}_1}}{e^{as}}$$

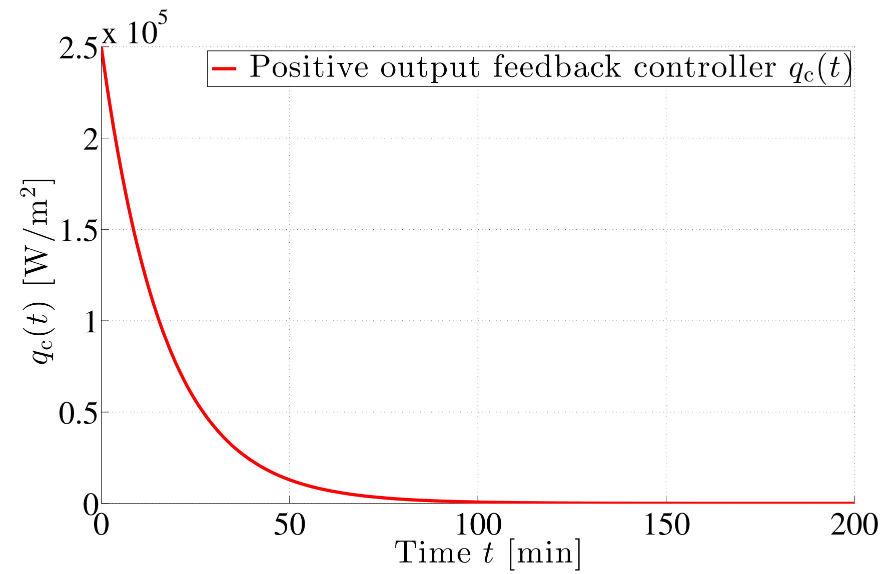
yields $\dot{W} \leq -bW$, which leads to

$$V \leq e^{a(s(t)-s_0)}V(0)e^{-bt} \underbrace{\leq}_{\because\, s_0 < s(t) < s_r} e^{a(s_r-s_0)}V(0)e^{-bt}$$

Numerical Simulation

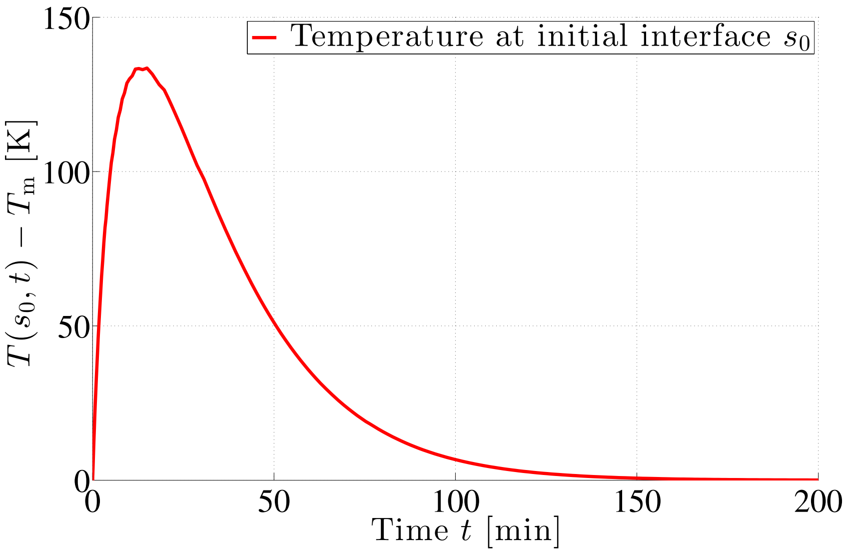
Zinc



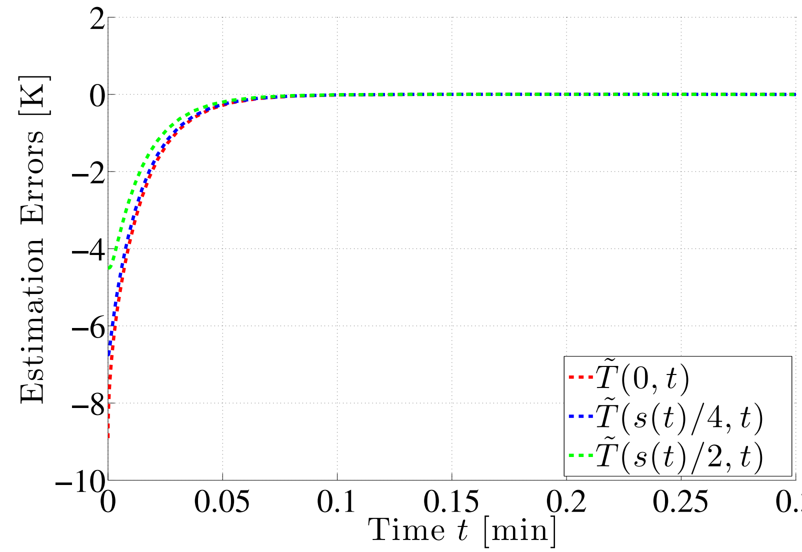


No overshoot

Positive heat



Temperature warms up and cool into melting point



Negative estimation errors