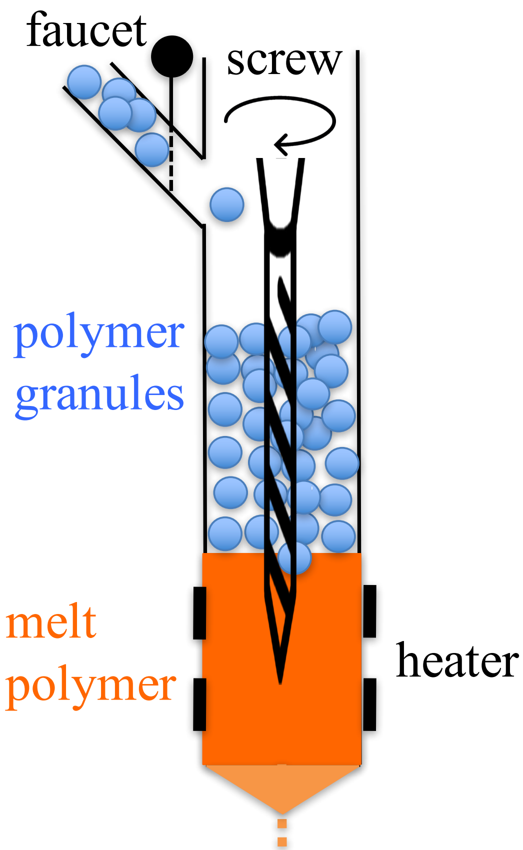
ISS for Control of Stefan Problem w.r.t Heat Loss at Interface

Shumon Koga, Iasson Karafyllis, Miroslav Krstic

University of California, San Diego

ACC 2018

Motivation : Screw Extruder for Polymer 3D Printing

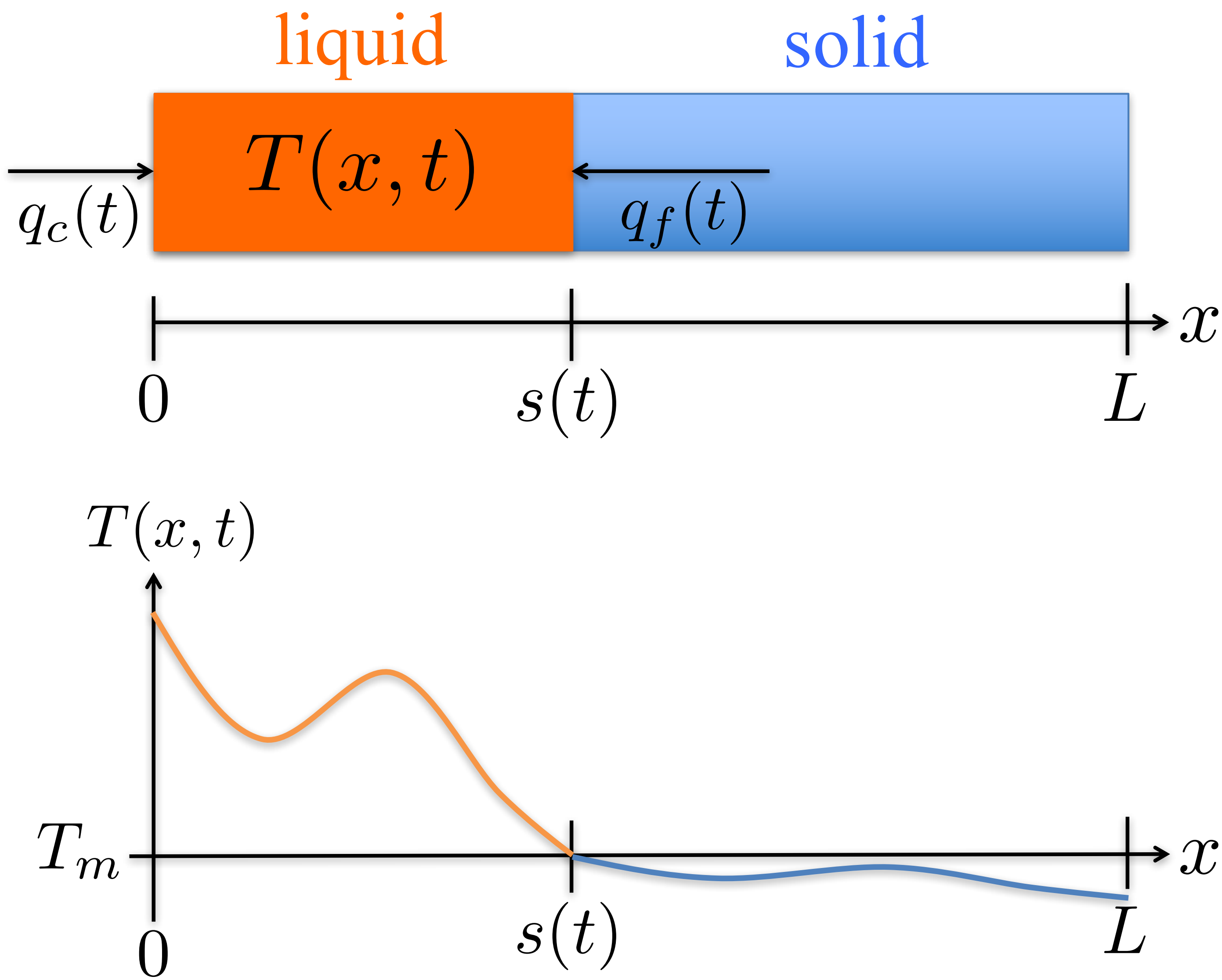


Model : Thermal phase change (melting/solidification)

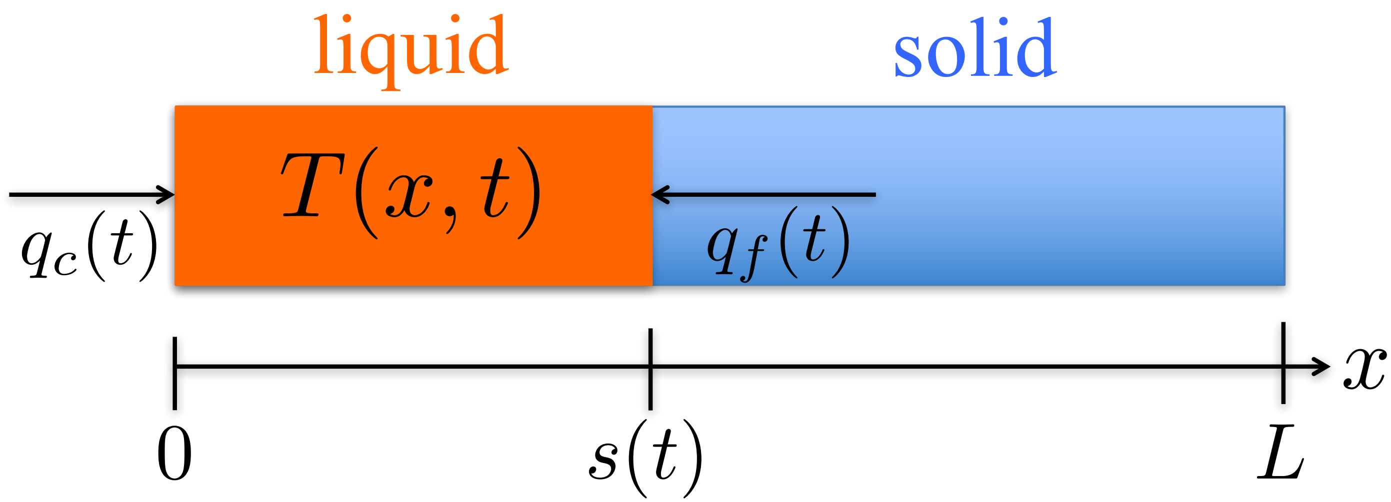
Objective : Stabilize ratio of granules/melt polymer

Property : Temperature in both phases are dynamic

Simplified Model : Melting + Heat Loss



Simplified Model : Melting + Heat Loss



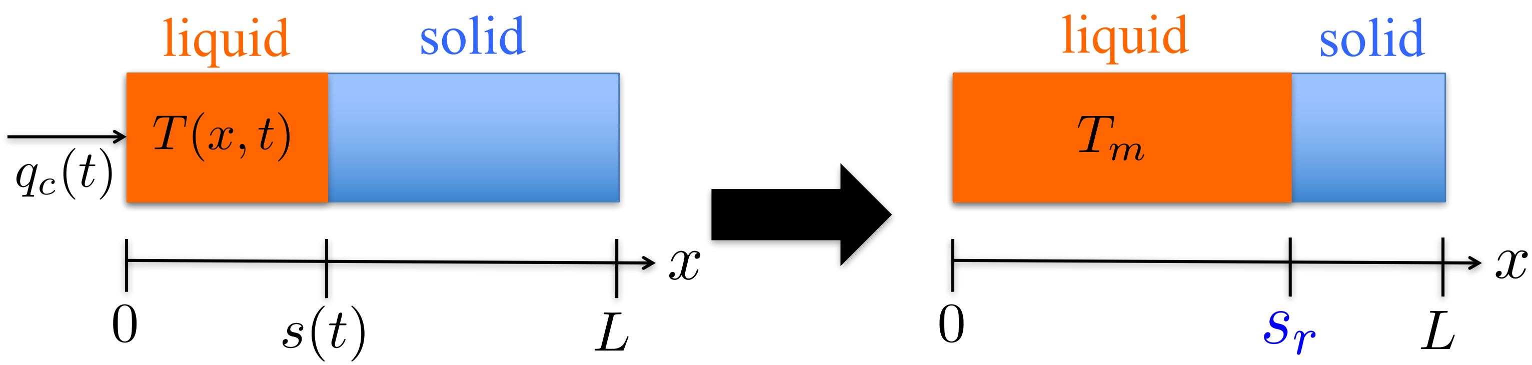
States : Temperature profile $T(x,t)$, Interface position $s(t)$

Control : Melting heat $q_c(t) > 0$

Disturbance : Freezing heat $q_f(t) > 0$ (magnitude) from solid phase

Problem

Previous work (ACC16) : Designed $q_c(t) > 0$ (feedback w.r.t. $T(x,t), s(t)$) to achieve $s(t) \to s_r$ under $q_f(t) \equiv 0$.



Problem

Previous work (ACC16) : Designed $q_c(t) > 0$ (feedback w.r.t. $T(x,t), s(t)$) to achieve $s(t) \to s_r$ under $q_f(t) \equiv 0$.

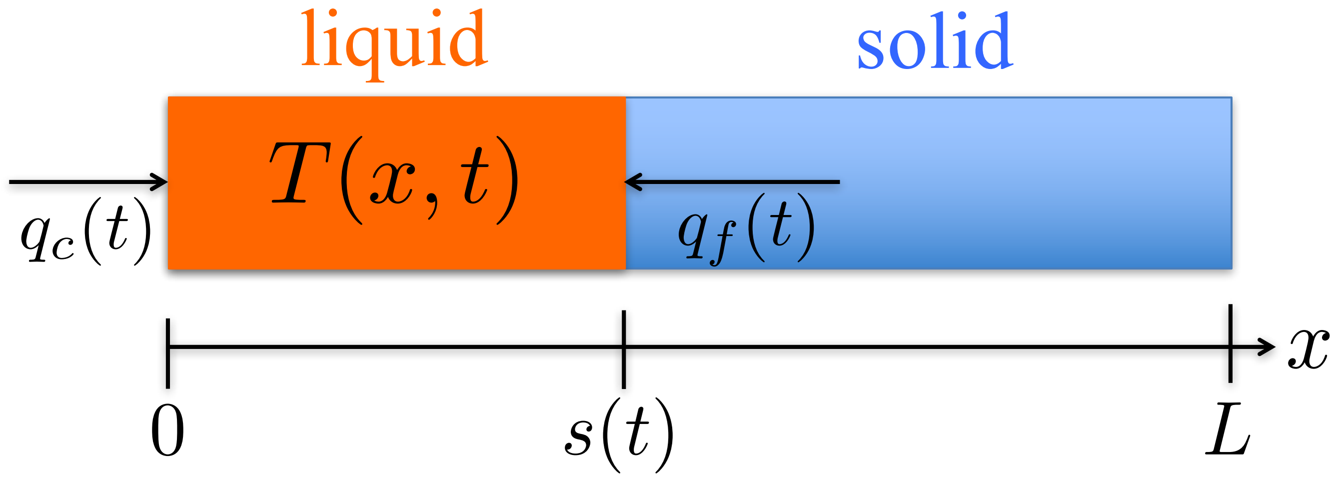
Question : What if there exists $q_f(t) \geq 0$? (under *same control design*)

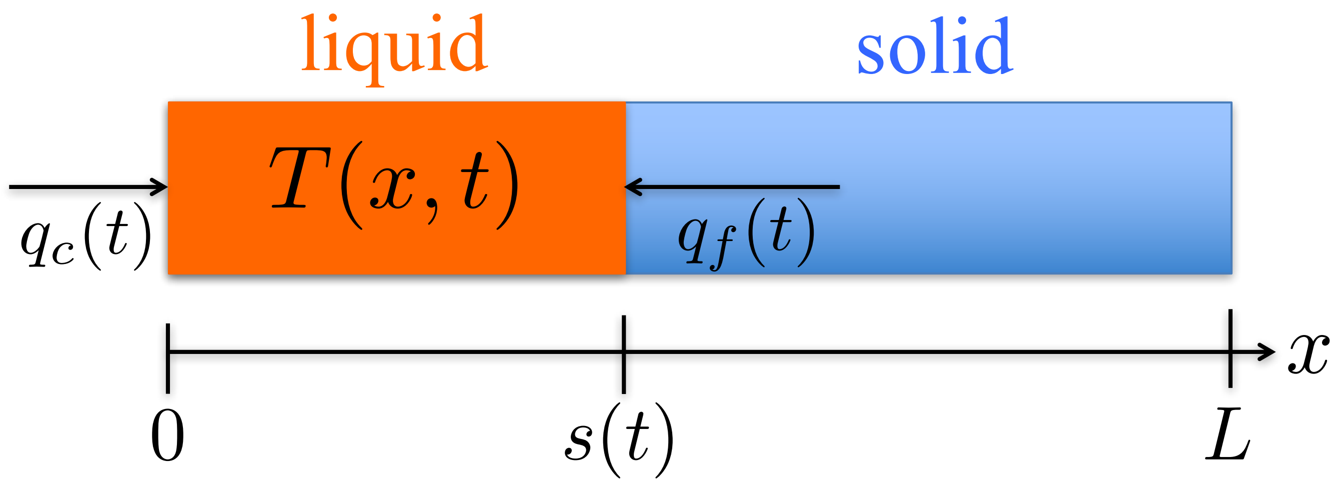
Problem

Previous work (ACC16) : Designed $q_c(t) > 0$ (feedback w.r.t. $T(x,t), s(t)$) to achieve $s(t) \to s_r$ under $q_f(t) \equiv 0$.

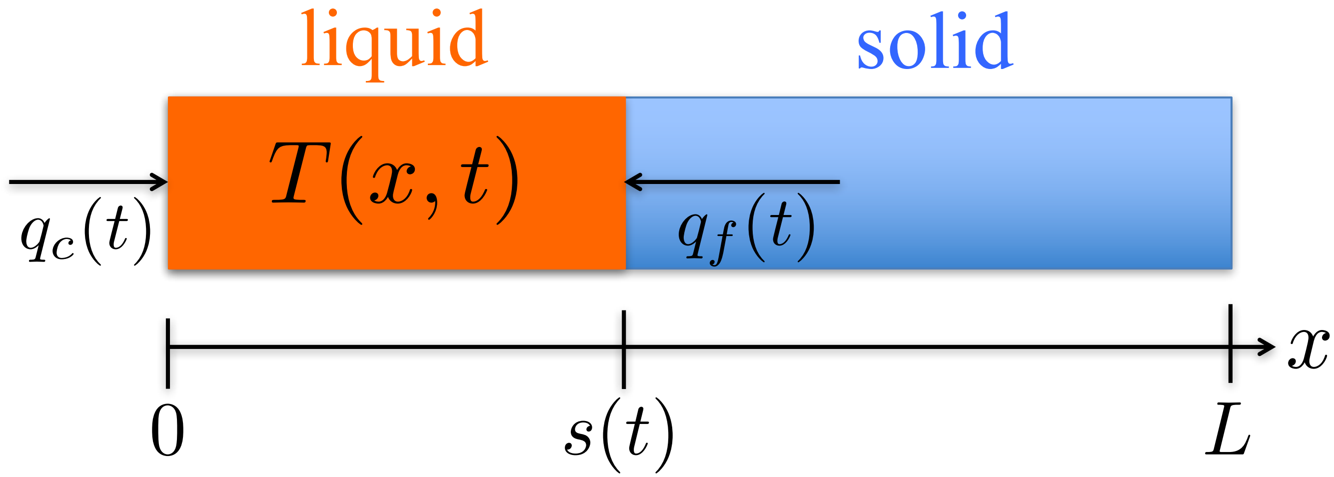
Question : What if there exists $q_f(t) \geq 0$? (under *same control design*)

$\rightarrow$ Prove Input-to-State Stability (ISS) w.r.t. $q_f(t)$


liquid
solid
$T(x,t)$
$q_c(t)$
$q_f(t)$
0
$s(t)$
L
x



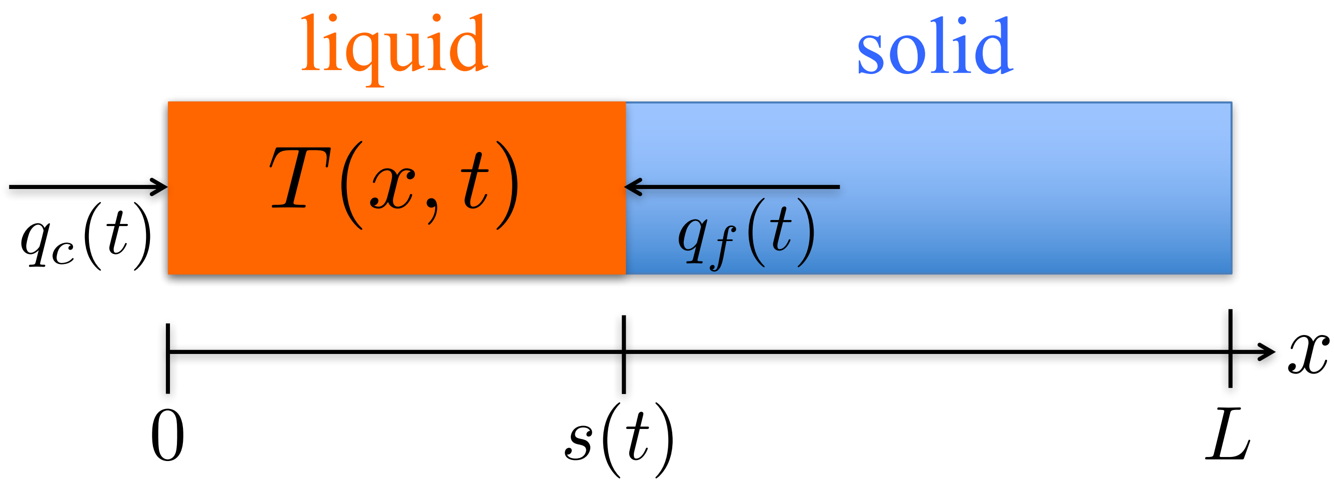
PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$



PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$

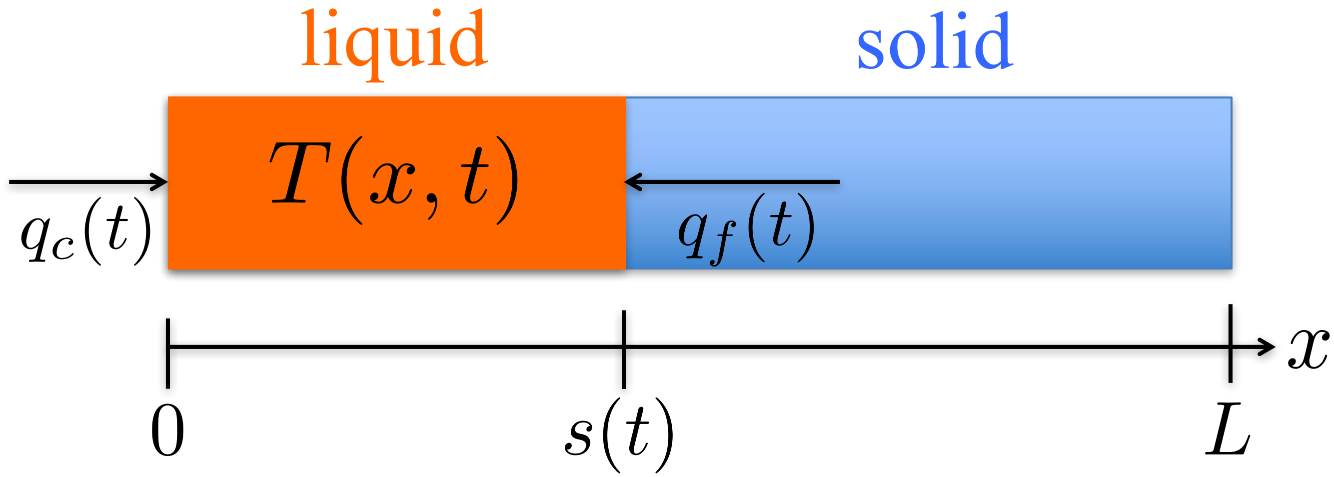


PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$

ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t) - \gamma q_f(t)$



PDE $\quad T_t(x,t) = \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L$

$$T_x(0,t) = -q_c(t)/k$$

$$T(s(t),t) = T_m$$

ODE $\quad \dot{s}(t) = -\beta T_x(s(t),t) - \gamma q_f(t)$

State-dependent moving boundary $\Rightarrow$ Nonlinear

Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$
$$0 < s(t) < L, \quad \forall t > 0$$

Lemma If $q_c(t) > 0$ $\forall t > 0$, then $T(x,t) > T_m,$ $\forall x \in (0, s(t)), \forall t > 0$

Model valid iff

$$T(x,t) > T_m, \quad \text{for} \quad \forall x \in (0, s(t)), \quad \forall t > 0$$
$$0 < s(t) < L, \quad \forall t > 0$$

Lemma If $q_c(t) > 0 \ \forall t > 0$, then $T(x,t) > T_m, \ \forall x \in (0, s(t)), \forall t > 0$

* $\boxed{q_c(t) > 0}$ and $0 < s(t) < L$ are proved after $q_c(t)$ is designed

Without heat loss ($q_f(t) \equiv 0$), the following assumption **necessary**

Assumption : Setpoint s_r chosen to satisfy

$$s_0 + \frac{\beta}{\alpha} \int_0^{s_0} (T_0(x) - T_m) dx < s_r < L$$

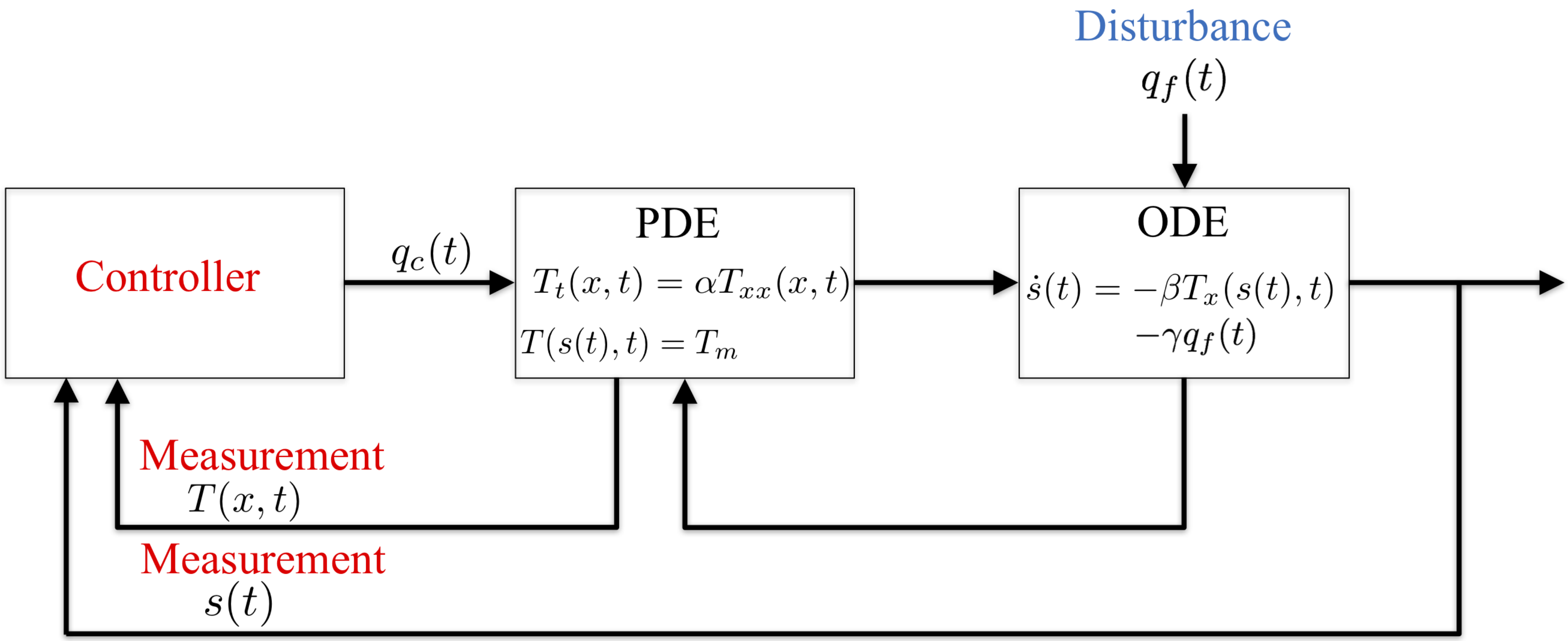
We impose the same assumption because $q_f(t)$ is an *unknown* disturbance.

Assumption : The heat loss is bounded and its total energy is also bounded, i.e., $\exists \bar{q}_f, M > 0$ s.t.

$$0 \leq q_f(t) \leq \bar{q}_f$$

,

$$\int_0^{\infty} q_f(t) dt \leq M$$


Disturbance
$q_f(t)$
Controller
$q_c(t)$
PDE
$T_t(x,t) = \alpha T_{xx}(x,t)$
$T(s(t),t) = T_m$
ODE
$\dot{s}(t) = -\beta T_x(s(t),t)$
$-\gamma q_f(t)$
Measurement
$T(x,t)$
Measurement
$s(t)$

Previous Result (ACC16) For $q_f(t) \equiv 0$, the control law

$$q_c(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t)} (T(x,t) - T_m)dx + \frac{k}{\beta}(s(t) - s_r)\right)$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm $\Psi(t) = ||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2$.

Previous Result (ACC16) For $q_f(t) \equiv 0$, the control law

$$q_c(t) = -c \left(\frac{k}{\alpha} \int_0^{s(t)} (T(x,t) - T_m)dx + \frac{k}{\beta}(s(t) - s_r) \right)$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm $\Psi(t) = ||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2$.

Theorem (ACC18) For $q_f(t) \geq 0$, the *same control law* with gain c satisfying $c > \frac{\beta}{k s_r}\bar{q}_{\mathrm{f}}$, makes the closed-loop system ISS w.r.t. $q_f(t)$ in the norm $\Psi(t) = ||T - T_m||^2_{\mathcal{L}_2} + (s - s_r)^2$.

Previous Result (ACC16) For $q_f(t) \equiv 0$, the control law

$$q_c(t) = -c\left(\frac{k}{\alpha}\int_0^{s(t)} (T(x,t) - T_m)dx + \frac{k}{\beta}(s(t) - s_r)\right)$$

where $c > 0$, makes the closed-loop system globally exponentially stable in the norm $\Psi(t) = ||T - T_m||^2_{\mathcal{H}_1} + (s - s_r)^2$.

Theorem (ACC18) For $q_f(t) \geq 0$, the *same control law* with gain c satisfying $c > \frac{\beta}{k s_r}\bar{q}_{\mathrm{f}}$, makes the closed-loop system ISS w.r.t. $q_f(t)$ in the norm $\Psi(t) = ||T - T_m||^2_{\mathcal{L}_2} + (s - s_r)^2$.

Note : Gain should be large to avoid complete frozen

Design Procedure

- Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

Design Procedure

- Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

- Backstepping transformation (from (u, X) to (w, X))

$$w(x,t) = u(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)} \phi(x-y)u(y,t)dy - \phi(x - s(t))X(t)$$

$$\phi(x) = \frac{c}{\beta}x - \varepsilon$$

Design Procedure

- Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

- Backstepping transformation (from (u, X) to (w, X))

$$w(x,t) = u(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)} \phi(x-y)u(y,t)dy - \phi(x-s(t))X(t)$$
$$\phi(x) = \frac{c}{\beta}x - \varepsilon$$

- Target system ($d(t) := \gamma q_f(t)$)

$$w_t(x,t) = \alpha w_{xx}(x,t) + \dot{s}(t)\phi'(x-s(t))X(t) + \phi(x-s(t))d(t),$$
$$w(s(t),t) = \varepsilon X(t) \quad w_x(0,t) = \frac{\beta}{\alpha}\phi(0)u(0)$$
$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t) - d(t)$$

$\rightarrow$ Stable if $d(t) \equiv 0$.

Design Procedure

- Reference errors

$$u(x,t) := T(x,t) - T_m, \quad X(t) := s(t) - s_r$$

- Backstepping transformation (from (u, X) to (w, X))

$$w(x,t) = u(x,t) - \frac{\beta}{\alpha}\int_x^{s(t)} \phi(x-y)u(y,t)dy - \phi(x - s(t))X(t)$$
$$\phi(x) = \frac{c}{\beta}x - \varepsilon$$

- Target system ($d(t) := \gamma q_f(t)$)

$$w_t(x,t) = \alpha w_{xx}(x,t) + \dot{s}(t)\phi'(x - s(t))X(t) + \phi(x - s(t))d(t),$$
$$w(s(t),t) = \varepsilon X(t) \quad w_x(0,t) = \frac{\beta}{\alpha}\phi(0)u(0)$$
$$\dot{X}(t) = -cX(t) - \beta w_x(s(t),t) - d(t)$$

$\rightarrow$ Stable if $d(t) \equiv 0$.

$\star$ The controller is derived by transformation & target system

Analysis

Lemma The closed-loop satisfies $\boxed{q_c(t) > 0}$ and $0 < s(t) < L$

Lemma Target (w, X)-system is ISS w.r.t. $d(t)$

Proof is by ISS Lyapunov function. Define

$$V = \frac{1}{2\alpha}||w||_{L_2}^2 + \frac{\varepsilon}{2\beta}X^2$$

and derive

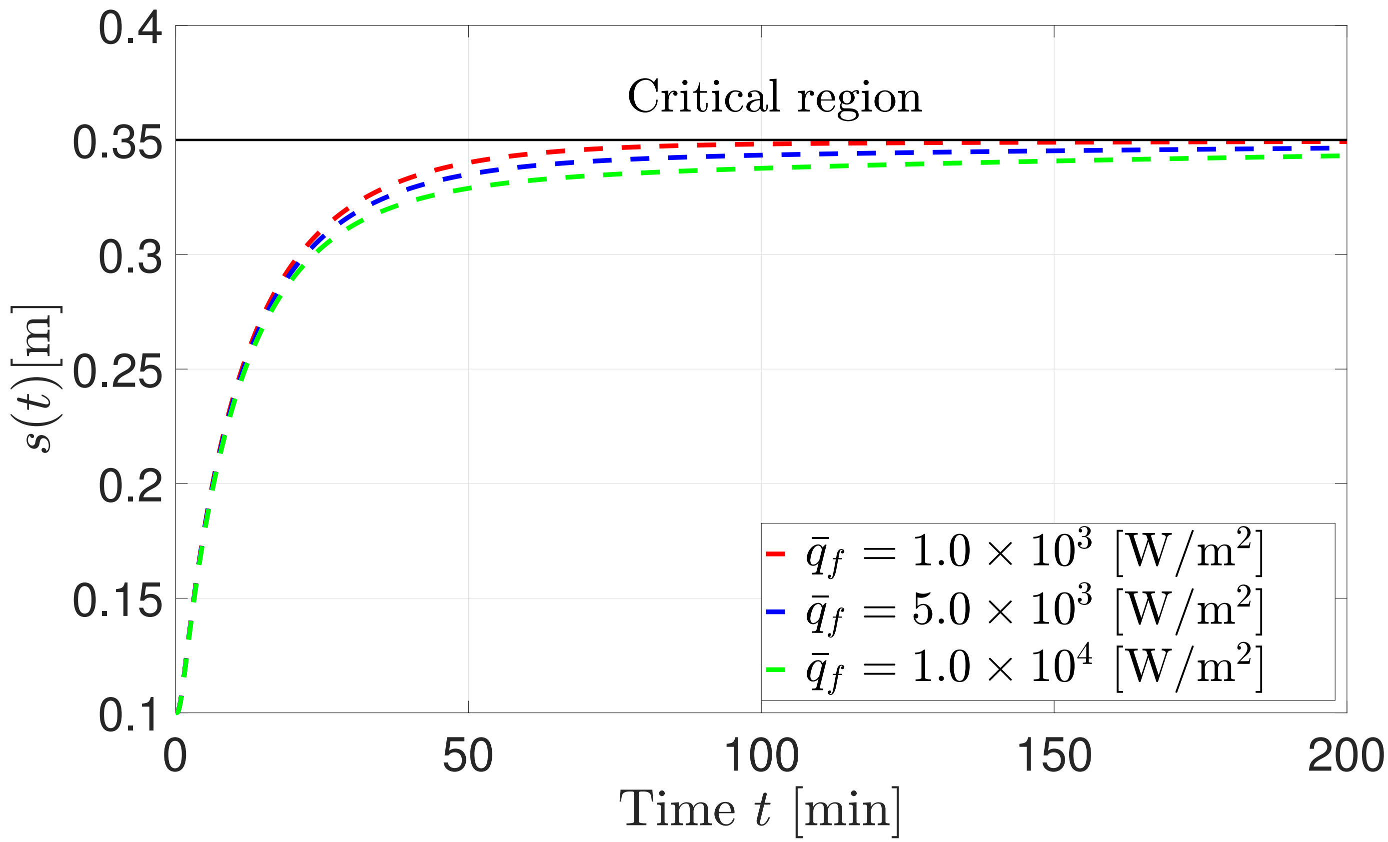
$$V(t) \leq M_1 V_0 e^{-bt} + M_2 \int_0^t e^{-b(t-\tau)} d(\tau)^2 d\tau$$

$\Rightarrow$ concludes ISS of (T, s) at (T_m, s_r) w.r.t q_f.

Numerical Simulation

Zinc

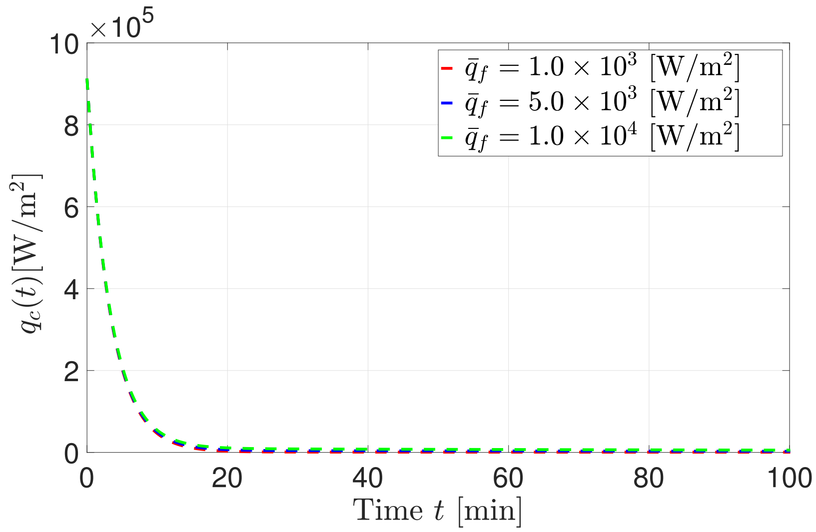
Heat loss $q_f(t) = \bar{q}_f e^{-Kt}$ with K *extremely small* (half life 40 [hour])

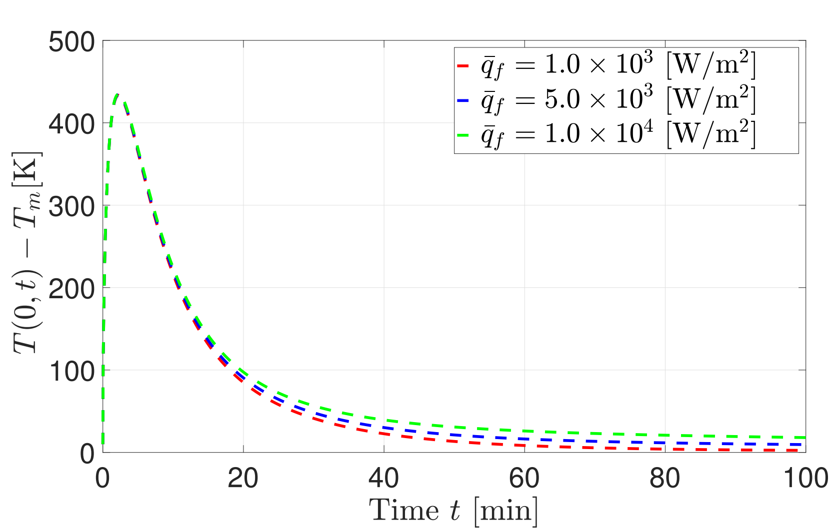


$\Rightarrow$ illustrates ISS property with $0 < s(t) < s_r$.

Numerical Simulation

Zinc





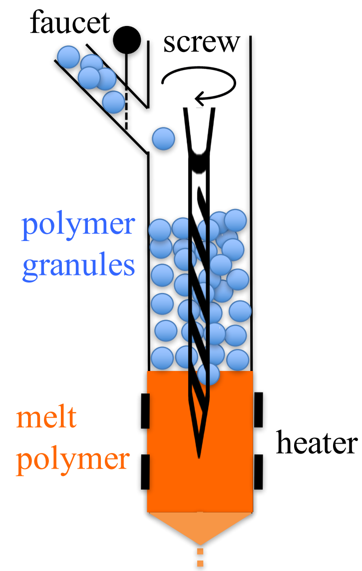
$\star$ heat input maintains positive,

$\star$ liquid temperature is above melting temperature

Future Work

- Incorporate screw extruder's dynamics

$$T_t = \alpha T_{xx} - bT_x - h(T - T_b)$$



- Redesign by two-phase temperature dynamics (CDC 2017)